DSm SUPER VECTOR SPACE OF REFINED LABELS

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PREFACE

In this book authors for the first time introduce the notion of supermatrices of refined labels. Authors prove super row matrix of refined labels form a group under addition. However super row matrix of refined labels do not form a group under product; it only forms a semigroup under multiplication. In this book super column matrix of refined labels and $m \times n$ matrix of refined labels are introduced and studied.

We mainly study this to introduce to super vector space of refined labels using matrices.

We in this book introduce the notion of semifield of refined labels using which we define for the first time the notion of supersemivector spaces of refined labels. Several interesting properties in this direction are defined and derived.

We suggest over hundred problems some of which are simple some at research level and some difficult. We give some applications but we are sure in due course when these new notions become popular among researchers they will find lots of applications.

This book has five chapters. First chapter is introductory in nature, second chapter introduces super matrices of refined labels and algebraic structures on these supermatrices of refined labels. All possible operations are on these supermatrices of refined labels is discussed in chapter three. Forth chapter introduces the notion of supermatrix of refined label vector spaces. Super matrix of refined labels of semivector spaces is introduced and studied and analysed in chapter five. Chapter six suggests the probable applications of these new structures. The final chapter suggests over hundred problems.

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W.B.VASANTHA KANDASAMY

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Chapter One

INTRODUCTION

This chapter has two sections. In section one we introduce the notion of super matrices and illustrate them by some examples. Using these super matrices super matrix labels are constructed in later chapters. Section two recalls the notion of ordinary labels, refined labels and partially ordered labels and illustrate them with examples. These concepts are essential to make this book a self contained one. For more refer [47-8].

1.1 Super Matrices and Their Properties

We call the usual matrix A = (a_{ij}) where $a_{ij} \in$ R or Q or Z or Z_n as a simple matrix. A simple matrix can be a square matrix like

$$A = \begin{bmatrix} 3 & 0 & 2 & 1 \\ -4 & 7 & -8 & 0 \\ 0 & -2 & 1 & 9 \\ \sqrt{2} & 0 & 1/4 & 0 \end{bmatrix}$$

or a rectangular matrix like

$$B = \begin{bmatrix} 8 & 0 & 7 & 1 & \sqrt{7} & \sqrt{3} \\ -2 & 5 & 8 & 0 & 9 & \sqrt{5} \\ 0 & 1 & 4 & -3 & 0 & \sqrt{7} \\ -7 & 2 & 0 & \sqrt{5} & \sqrt{2} & 0 \\ 1 & 0 & 1 & -2 & 1 & 8 \end{bmatrix}$$

or a row matrix like T = (9, $\sqrt{2}$, -7, 3, -5, $-\sqrt{7}$, 8, 0, 1, 2) or a column matrix like

$$P = \begin{bmatrix} 2 \\ 0 \\ \sqrt{7} \\ -8 \\ 3 \\ 2 \\ \sqrt{51} \\ 9 \end{bmatrix}.$$

So one can define a super matrix as a matrix whose elements are submatrices. For instance

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

where

$$a_{11} = \begin{bmatrix} 9 & 0 \\ 1 & 2 \\ 5 & -7 \end{bmatrix}, a_{12} = \begin{bmatrix} 3 & 7 & 8 \\ -1 & 0 & 9 \\ 2 & 7 & -6 \end{bmatrix}, a_{21} = \begin{bmatrix} 8 & -4 \\ 4 & 5 \\ 1 & 2 \end{bmatrix},$$

$$a_{31} = \begin{bmatrix} 9 & 3 \\ -1 & 2 \\ \sqrt{3} & -7 \\ -9 & 0 \\ 0 & 8 \end{bmatrix}, a_{22} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 0 & 8 & -1 \end{bmatrix} \text{ and } a_{32} = \begin{bmatrix} 9 & 2 & 0 \\ 0 & 1 & 2 \\ 3 & 0 & 4 \\ 0 & 5 & 7 \\ 8 & 0 & 0 \end{bmatrix}$$

are submatrices. So submatrices are associated with a super matrix [47]. The height of a super matrix is the number of rows of submatrices in it and the width of a super matrix is the number of columns of submatrices in it [47].

We obtain super matrix from a simple matrix. This process of constructing super matrix from a simple matrix will be known as the partitioning. A simple matrix is partitioned by dividing or separating the simple matrix of a specified row and specified column. When division is carried out only between the columns then those matrices are called as super row vectors; when only rows are partitioned in simple matrices we call those simple matrices as super column vectors.

We will first illustrate this situation by some simple examples.

***Example 1.1.1*:** Let

$$A = \begin{bmatrix} 0 & 7 & 5 & 7 & 8 & 4 & 2 & 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 \\ 5 & 1 & 3 & 1 & 1 & 1 & 2 & 1 & 3 & 1 & 4 \end{bmatrix}$$

be a 3×11 simple matrix. The super matrix or the super row vector is obtained by partitioning between the columns.

$$A_1 = \left[\begin{array}{cc|cccc|cc|ccc} 0 & 7 & 5 & 7 & 8 & 4 & 2 & 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 \\ 5 & 1 & 3 & 1 & 1 & 1 & 2 & 1 & 3 & 1 & 4 \end{array}\right]$$

is a super row vector.

$$A_2 = \left[\begin{array}{cc|ccccc|ccc|c} 0 & 7 & 5 & 7 & 8 & 4 & 2 & 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 \\ 5 & 1 & 3 & 1 & 1 & 1 & 2 & 1 & 3 & 1 & 4 \end{array}\right]$$

is a super row vector.

We can get several super row vectors from one matrix A.

***Example 1.1.2*:** Let V = (1 2 3 4 5 6 7 8 9 1 0 9 4 7 2 4 3) be a simple row vector we get a super row vector V_1 = (1 2 3 | 4 5 6 | 7 8 9 1 0 | 9 4 7 | 2 4 3) and many more super row vectors can be found by partitioning V differently.

Example 1.1.3: Let

$$V = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 5 & 7 \\ 1 & 2 & 3 \\ 4 & 5 & 7 \\ 8 & 9 & 0 \\ 1 & 0 & 7 \\ 0 & 1 & 4 \end{bmatrix}$$

be a simple matrix. We get the super column vector by partitioning in between the columns as follows:

$$V_1 = \left[\begin{array}{ccc} 3 & 0 & 1 \\ 0 & 5 & 7 \\ 1 & 2 & 3 \\ \hline 4 & 5 & 7 \\ \hline 8 & 9 & 0 \\ 1 & 0 & 7 \\ \hline 0 & 1 & 4 \end{array}\right] \text{ and } V_2 = \left[\begin{array}{ccc} 3 & 0 & 1 \\ \hline 0 & 5 & 7 \\ 1 & 2 & 3 \\ 4 & 5 & 7 \\ \hline 8 & 9 & 0 \\ 1 & 0 & 7 \\ 0 & 1 & 4 \end{array}\right]$$

are super column vectors.

***Example 1.1.4*:** Let

$$X = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 0 \end{bmatrix}$$

be a simple matrix.

$$X' = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ \hline 5 \\ 6 \\ \hline 7 \\ 8 \\ \hline 9 \\ 0 \end{bmatrix}$$

is a super column vector. We can get several such super column vectors using X.

Example 1.1.5: Let us consider the simple matrix;

$$P = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 4 & -8 & -4 & 7 & 3 \\ 2 & 5 & 1 & -5 & 8 & 2 \end{bmatrix},$$

we get the super matrix by partitioning P as follows:

$$P_1 = \left[\begin{array}{ccc|c|cc} 0 & 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 & 11 \\ \hline 1 & 4 & -8 & -4 & 7 & 3 \\ 2 & 5 & 1 & -5 & 8 & 2 \end{array}\right]$$

is a super matrix with 6 submatrices.

$$a_1 = \begin{bmatrix} 0 & 1 & 2 \\ 6 & 7 & 8 \end{bmatrix},\ a_2 = \begin{bmatrix} 3 \\ 9 \end{bmatrix},\ a_3 = \begin{bmatrix} 4 & 5 \\ 10 & 11 \end{bmatrix},$$

$$a_4 = \begin{bmatrix} 1 & 4 & -8 \\ 2 & 5 & 1 \end{bmatrix},\ a_5 = \begin{bmatrix} -4 \\ -5 \end{bmatrix} \text{ and } a_6 = \begin{bmatrix} 7 & 3 \\ 8 & 2 \end{bmatrix}.$$

Thus $P_1 = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}$. Clearly P_1 is a 4×6 matrix.

Example 1.1.6: Let

$$V = \left[\begin{array}{cc|cc} 3 & 6 & 3 & 1 \\ 1 & -3 & 8 & 2 \\ \hline 2 & 4 & 9 & 3 \\ 7 & -1 & 4 & 4 \end{array}\right]$$

be a super matrix.

$$V = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

where a_1, a_2, a_3 and a_4 are submatrices with

$$a_1 = \begin{bmatrix} 3 & 6 \\ 1 & -3 \end{bmatrix},\ a_2 = \begin{bmatrix} 3 & 1 \\ 8 & 2 \end{bmatrix},\ a_3 = \begin{bmatrix} 2 & 4 \\ 7 & -1 \end{bmatrix} \text{ and } a_4 = \begin{bmatrix} 9 & 3 \\ 4 & 4 \end{bmatrix}.$$

We can also have the notion of super identity matrix as follows.

Let

$$S = \begin{bmatrix} I_{t_1} & & & \\ & I_{t_2} & 0 & \\ & & \ddots & \\ & 0 & & I_{t_r} \end{bmatrix}$$

where I_{t_i} are identity matrices of order $t_i \times t_i$ where $1 \leq i \leq r$.

We will just illustrate this situation by some examples.

Consider

$$P = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} I_4 & 0 \\ 0 & I_3 \end{bmatrix}$$

is a identity super diagonal matrix.

Consider

$$S = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} I_2 & 0 & 0 \\ 0 & I_3 & 0 \\ 0 & 0 & I_2 \end{bmatrix}$$

is the super diagonal identity matrix [47].

We now proceed onto recall the notion of general diagonal super matrices. Let

$$S = \left[\begin{array}{cc|cccc} 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 7 & 0 & 0 & 0 & 0 \\ 8 & 9 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 4 & 5 & 3 & 2 \\ 0 & 0 & -1 & 0 & 7 & -4 \end{array}\right] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

where

$$m_1 = \begin{bmatrix} 3 & 0 \\ 1 & 7 \\ 8 & 9 \\ 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ amd } m_2 = \begin{bmatrix} 4 & 5 & 3 & 2 \\ -1 & 0 & 7 & -4 \end{bmatrix},$$

S is a general diagonal super matrix. Take

$$K = \left[\begin{array}{cccccc|cccc} 3 & 4 & 5 & 7 & 8 & 4 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 5 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 9 & 6 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 7 & 8 & 1 \end{array}\right] = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$$

is a general diagonal super matrix.

Also take

$$V = \left[\begin{array}{c|c|c|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

is again a general diagonal super matrix.

Now [47] defines also the concept of partial triangular matrix as a super matrix.

Consider

$$S = \left[\begin{array}{cccc|ccc} 6 & 2 & 1 & 4 & 6 & 2 & 1 \\ 0 & 7 & 8 & 9 & 0 & 3 & 4 \\ 0 & 0 & 1 & 4 & 4 & 0 & 5 \\ 0 & 0 & 0 & 3 & 7 & 8 & 9 \end{array}\right] = [M_1\ M_2]$$

is a upper partial triangular super matrix.

Suppose

$$P = \left[\begin{array}{ccccc} 2 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ 2 & 3 & 5 & 0 & 0 \\ 1 & 2 & 3 & 4 & 0 \\ 5 & 4 & 7 & 8 & 0 \\ \hline 8 & 0 & 1 & 2 & 3 \\ 1 & 1 & 0 & 2 & 4 \\ 0 & 1 & 2 & 0 & 3 \end{array}\right] = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$$

is a lower partial triangular matrix. This is a 2 by 1 super matrix where the first element in this super matrix is a square lower triangular matrix and the second element is a rectangular matrix. We have seen column super vectors and row super vectors.

Now we can define as in case of simple matrices the concept of transpose of a super matrix. We will only indicate the situation by some examples, however for more refer [47]. Consider

$$V = \begin{bmatrix} 9 \\ 3 \\ 1 \\ \hline 0 \\ 2 \\ 4 \\ 6 \\ \hline 7 \\ 2 \end{bmatrix}$$

be the super column vector. Now transpose of V denoted by V^t = [9 3 1 | 0 2 4 6 | 7 2]. Clearly V^t is a super row vector. Thus if

$$V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

then

$$V^t = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}^t = \begin{bmatrix} v_1^t \\ v_2^t \\ v_3^t \end{bmatrix} = \begin{bmatrix} v_1^t & v_2^t & v_3^t \end{bmatrix}.$$

For $v_1 = \begin{bmatrix} 9 \\ 3 \\ 1 \end{bmatrix}$ and v_1^t = (9 3 1), $v_2 = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 6 \end{bmatrix}$ and

v_2^t = (0 2 4 6) and $v_3 = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$ and v_3^t = (7 2).

Hence the claim.

Example 1.1.7: Let P = [3 1 0 2 5 | 7 3 2 | 6] be a super row vector. Now transpose of P denoted by

$$P^t = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 5 \\ \hline 7 \\ 3 \\ 2 \\ \hline 6 \end{bmatrix}.$$

Thus if $P = [P_1\ P_2\ P_3]$ where

$$P_1 = (3\ 1\ 0\ 2\ 5),\ P_2 = (7\ 3\ 2) \text{ and } P_3 = (6)$$

then

$$P_1^t = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 5 \end{bmatrix},\ P_2^t = \begin{bmatrix} 7 \\ 3 \\ 2 \end{bmatrix} \text{ and } P_3^t = 6.$$

Thus

$$P^t = \begin{bmatrix} P_1^t \\ P_2^t \\ P_3^t \end{bmatrix}.$$

Example 1.1.8: Consider the super row vector

$$X = \left[\begin{array}{ccc|ccc} 8 & 1 & 0 & 3 & 9 & 2 \\ 0 & 2 & 1 & 6 & 4 & 5 \\ 0 & 0 & 4 & 1 & 0 & 1 \end{array}\right].$$

Now define the transpose of X.

$$X^t = \left[\begin{array}{ccc|ccc} 8 & 1 & 0 & 3 & 9 & 2 \\ 0 & 2 & 1 & 6 & 4 & 5 \\ 0 & 0 & 4 & 1 & 0 & 1 \end{array}\right]^t = [A_1\ A_2] = \left[\begin{array}{ccc} 8 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 4 \\ \hline 3 & 6 & 1 \\ 9 & 4 & 0 \\ 2 & 5 & 1 \end{array}\right] = \begin{bmatrix} A_1^t \\ A_2^t \end{bmatrix}.$$

Example 1.1.9: Let

$$W = \left[\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 0 & 7 & 8 \\ 9 & 0 & 1 & 0 & 2 \\ 3 & 1 & 4 & 0 & 3 \\ \hline 1 & 6 & 0 & 5 & 1 \\ 2 & 7 & 1 & 0 & 7 \\ 3 & 8 & 0 & 2 & 0 \\ 4 & 9 & 1 & 2 & 4 \\ 5 & 0 & 4 & 0 & 3 \end{array}\right] = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$$

be a super column vector.

Now the transpose of W given by

$$W^t = \left[\begin{array}{cccc|ccccc} 0 & 5 & 9 & 3 & 1 & 2 & 3 & 4 & 5 \\ 1 & 6 & 0 & 1 & 6 & 7 & 8 & 9 & 0 \\ 2 & 0 & 1 & 4 & 0 & 1 & 0 & 1 & 4 \\ 3 & 7 & 0 & 0 & 5 & 0 & 2 & 2 & 0 \\ 4 & 8 & 2 & 3 & 1 & 7 & 0 & 4 & 3 \end{array}\right] = \begin{bmatrix} W_1^t & W_2^t \end{bmatrix}.$$

Now having see examples of transpose of a super row vector and super column vector we now proceed onto define the transpose of a super matrix.

Example 1.1.10: Let

$$M = \left[\begin{array}{cc|ccc|cc} 9 & 0 & 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 0 & 1 & 1 \\ \hline 1 & 2 & 1 & 3 & 1 & 4 & 1 \\ 5 & 1 & 6 & 1 & 7 & 1 & 8 \\ 1 & 9 & 2 & 0 & 2 & 1 & 2 \\ \hline 2 & 0 & 0 & 3 & 4 & 0 & 1 \end{array}\right] = \begin{bmatrix} M_1 & M_2 & M_3 \\ M_4 & M_5 & M_6 \\ M_7 & M_8 & M_9 \end{bmatrix}$$

be a super matrix, where

$$M_1 = \begin{bmatrix} 9 & 0 \\ 6 & 7 \end{bmatrix},\ M_2 = \begin{bmatrix} 1 & 2 & 3 \\ 8 & 9 & 0 \end{bmatrix},\ M_3 = \begin{bmatrix} 4 & 5 \\ 1 & 1 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} 1 & 2 \\ 5 & 1 \\ 1 & 9 \end{bmatrix},\ M_5 = \begin{bmatrix} 1 & 3 & 1 \\ 6 & 1 & 7 \\ 2 & 0 & 2 \end{bmatrix},\ M_6 = \begin{bmatrix} 4 & 1 \\ 1 & 8 \\ 1 & 2 \end{bmatrix},$$

$$M_7 = (2, 0),\ M_8 = (0, 3, 4) \text{ and } M_9 = [0\ 1].$$

Consider

$$M^t = \left[\begin{array}{cc|ccc|c} 9 & 6 & 1 & 5 & 1 & 2 \\ 0 & 7 & 2 & 1 & 9 & 0 \\ \hline 1 & 8 & 1 & 6 & 2 & 0 \\ 2 & 9 & 3 & 1 & 0 & 3 \\ 3 & 0 & 1 & 7 & 2 & 4 \\ \hline 4 & 1 & 4 & 1 & 1 & 0 \\ 5 & 1 & 1 & 8 & 2 & 1 \end{array}\right] = \begin{bmatrix} M_1^t & M_4^t & M_7^t \\ M_2^t & M_5^t & M_8^t \\ M_3^t & M_6^t & M_9^t \end{bmatrix}$$

where

$$M_1^t = \begin{bmatrix} 9 & 6 \\ 0 & 7 \end{bmatrix},\ M_2^t = \begin{bmatrix} 1 & 8 \\ 2 & 9 \\ 3 & 0 \end{bmatrix},\ M_3^t = \begin{bmatrix} 4 & 1 \\ 5 & 1 \end{bmatrix},$$

$$M_4^t = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 9 \end{bmatrix},\ M_5^t = \begin{bmatrix} 1 & 6 & 2 \\ 3 & 1 & 0 \\ 1 & 7 & 2 \end{bmatrix},\ M_6^t = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 8 & 2 \end{bmatrix}$$

$$M_7^t = \begin{bmatrix} 2 \\ 0 \end{bmatrix},\ M_8^t = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \text{ and } M_9^t = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

M^t is the transpose of the super matrix M.

Example 1.1.11: Let

$$M = \left[\begin{array}{ccc|cc} 1 & 9 & 0 & 2 & 7 \\ 3 & 0 & 4 & 0 & 5 \\ \hline 6 & 7 & 8 & 9 & 0 \\ 2 & 3 & 4 & 0 & 5 \\ 0 & 1 & 0 & 2 & 1 \\ 5 & 0 & 8 & 0 & 9 \\ \hline 6 & 1 & 0 & 8 & 0 \end{array}\right] = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \\ M_5 & M_6 \end{bmatrix}$$

be the super matrix where

$$M_1 = \begin{bmatrix} 1 & 9 & 0 \\ 3 & 0 & 4 \end{bmatrix},\ M_2 = \begin{bmatrix} 2 & 7 \\ 0 & 5 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 6 & 7 & 8 \\ 2 & 3 & 4 \\ 0 & 1 & 0 \\ 5 & 0 & 8 \end{bmatrix},\ M_4 = \begin{bmatrix} 9 & 0 \\ 0 & 5 \\ 2 & 1 \\ 0 & 9 \end{bmatrix},$$

$M_5 = (6\ 1\ 0)$ and $M_6 = (8, 0)$. Consider M^t, the transpose of M.

$$M^t = \left[\begin{array}{cc|cccc|c} 1 & 3 & 6 & 2 & 0 & 5 & 6 \\ 9 & 0 & 7 & 3 & 1 & 0 & 1 \\ 0 & 4 & 8 & 4 & 0 & 8 & 0 \\ \hline 2 & 0 & 9 & 0 & 2 & 0 & 8 \\ 7 & 5 & 0 & 5 & 1 & 9 & 0 \end{array}\right] = \begin{bmatrix} M_1^t & M_3^t & M_5^t \\ M_2^t & M_4^t & M_6^t \end{bmatrix}$$

where

$$M_1^t = \begin{bmatrix} 1 & 3 \\ 9 & 0 \\ 0 & 4 \end{bmatrix}\ M_2^t = \begin{bmatrix} 2 & 0 \\ 7 & 5 \end{bmatrix},\ M_3^t = \begin{bmatrix} 6 & 2 & 0 & 5 \\ 7 & 3 & 1 & 0 \\ 8 & 4 & 0 & 8 \end{bmatrix},$$

$$M_4^t = \begin{bmatrix} 9 & 0 & 2 & 0 \\ 0 & 5 & 1 & 9 \end{bmatrix},\ M_5^t = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix},\ M_6^t = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$$

is the transpose of the super matrix M.

We say two $1 \times m$ super row vectors are similar if and only if have identical partition. That is partitioned in the same way.

Consider X = (0 1 2 | 3 4 | 5 0 9) and Y = (3 0 5 | 2 1 | 8 1 2) two similar super row vectors. However Z = (0 | 1 2 3 | 4 5 0 | 9) is not similar with X or Y as it has a different partition though X = Z = (0 1 2 3 4 5 0 9) as simple matrices. Likewise we can say two $n \times 1$ super column vectors are similar if they have identical partition in them.

For consider

$$X = \left[\begin{array}{c} 9 \\ 8 \\ 7 \\ \hline 0 \\ 1 \\ \hline 2 \\ 3 \\ 4 \\ 5 \end{array}\right] \text{ and } Y = \left[\begin{array}{c} 2 \\ 1 \\ 0 \\ \hline 3 \\ 7 \\ \hline 8 \\ 0 \\ 1 \\ 2 \end{array}\right]$$

two super column vectors. Clearly X and Y are similar. However if

$$Z = \left[\begin{array}{c} 8 \\ \hline 10 \\ 2 \\ \hline 3 \\ 1 \\ \hline 4 \\ 9 \\ \hline 0 \\ 2 \end{array}\right]$$

is a 9 × 1 simple column matrix but X and Y are not similar with Z as Z has a different partition.

We now proceed onto give examples of similar super matrices.

Example 1.1.12: Let

$$X = \left[\begin{array}{cccc} 3 & 7 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ \hline 2 & 0 & 1 & 0 \\ 0 & 3 & 0 & 4 \\ 4 & 0 & 1 & 0 \\ \hline 5 & 1 & 2 & 1 \\ 3 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 4 & 4 & 4 & 4 \\ \hline 0 & 5 & 5 & 0 \end{array}\right] \text{ and } Y = \left[\begin{array}{cccc} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ \hline 8 & 9 & 1 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 3 & 1 & 4 \\ \hline 1 & 5 & 1 & 6 \\ 1 & 7 & 1 & 8 \\ 1 & 9 & 2 & 0 \\ 2 & 0 & 2 & 2 \\ \hline 2 & 3 & 2 & 4 \end{array}\right]$$

be two super column vectors. Clearly X and Y are similar super column vectors.

Consider

$$P = \left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \\ \hline 5 & 6 \\ 7 & 8 \\ 9 & 1 \\ \hline 1 & 2 \\ 1 & 3 \\ 5 & 1 \\ 7 & 0 \\ \hline 9 & 2 \end{array}\right]$$

a super column vector. Though both P and Y have similar partition yet P and Y are not similar super matrices.

Thus if P and Y are similar in the first place they must have same natural order and secondly the partitions on them must be identical.

Example 1.1.13: Let

$$X = \left[\begin{array}{ccc|cc|cc|ccc|c} 2 & 3 & 7 & 0 & 2 & 5 & 7 & 9 & 0 & 1 & 4 \\ 0 & 5 & 8 & 1 & 3 & 6 & 1 & 0 & 8 & 2 & 5 \\ 1 & 6 & 9 & 1 & 1 & 1 & 8 & 1 & 3 & 3 & 6 \end{array}\right]$$

be a super row vector.

Take

$$Y = \left[\begin{array}{ccc|cc|cc|ccc|c} 1 & 3 & 0 & 1 & 8 & 9 & 1 & 7 & 6 & 3 & 1 \\ 0 & 4 & 0 & 0 & 1 & 7 & 0 & 4 & 0 & 9 & 2 \\ 2 & 5 & 7 & 0 & 2 & 8 & 2 & 2 & 1 & 1 & 3 \end{array}\right]$$

be another super row vector. Clearly X and Y are similar super row matrices. That is X and Y have identical partition on them.

Example 1.1.14: Let

$$X = \left[\begin{array}{ccc|cc} 3 & 1 & 2 & 5 & 4 \\ 7 & 8 & 9 & 0 & 1 \\ \hline 1 & -1 & 0 & 3 & 8 \\ 2 & 0 & 4 & 0 & 9 \\ 3 & 2 & -1 & 0 & 1 \\ 1 & 0 & 2 & -4 & 2 \end{array}\right]$$

and

$$Y = \left[\begin{array}{ccc|cc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 7 & 9 & 0 \\ \hline 1 & 1 & 1 & 2 & 3 \\ -1 & 4 & 5 & -1 & 0 \\ 7 & 0 & 0 & 0 & 7 \\ 0 & 1 & -2 & 3 & 1 \end{array}\right]$$

be two super matrices.

Both X and Y enjoy the same or identical partition hence X and Y are similar super matrices.

Example 1.1.15: Let

$$X = \left[\begin{array}{cc|ccc}
1 & 2 & 3 & 4 & 5 \\
3 & 0 & 1 & 0 & 2 \\
\hline
7 & 8 & 0 & -1 & 2 \\
-1 & 4 & -1 & 0 & -2 \\
0 & -5 & 1 & 0 & 1 \\
1 & 2 & -1 & 4 & 0 \\
\hline
0 & 1 & 2 & 0 & 1 \\
1 & 0 & 0 & 1 & 2 \\
-2 & 4 & 3 & 4 & -1
\end{array}\right]$$

and

$$Y = \left[\begin{array}{cc|ccc}
4 & 0 & 2 & 1 & 5 \\
0 & -2 & 1 & 0 & 2 \\
\hline
7 & -1 & 1 & 6 & 2 \\
0 & 8 & 0 & -7 & 0 \\
1 & -4 & 8 & 1 & 6 \\
8 & 0 & 0 & 9 & 1 \\
\hline
3 & 1 & 7 & 0 & 2 \\
2 & -1 & 6 & 4 & 0 \\
0 & 4 & 0 & 2 & 1
\end{array}\right]$$

be two super matrices.

Clearly X and Y are similar super matrices as they enjoy identical partition. We can add only when two super matrices of same natural order and enjoy similar partition otherwise addition is not defined on them.

We will just show how addition is performed on super matrices.

Example 1.1.16: Let $X = (0\ 1\ 2 \mid 1\ {-3}\ 4\ 5 \mid {-7}\ 8\ 9\ 0\ {-1})$ and $Y = (8\ {-1}\ 3 \mid {-1}\ 2\ 3\ 4 \mid 8\ 1\ 0\ 1\ 2)$ be any two similar super row vectors. Now we can add X with Y denoted by $X + Y = (0\ 1\ 2 \mid 1\ {-3}\ 4\ 5 \mid {-7}\ 8\ 9\ 0\ {-1}) + (8\ {-1}\ 3 \mid {-1}\ 2\ 3\ 4 \mid 8\ 1\ 0\ 1\ 2) = (8\ 0\ 5 \mid 0\ 0\ {-1}\ 7\ 9 \mid 1\ 9\ 9\ 1\ 1)$. We say X + Y is also a super row vector which is similar with X and Y.

Now we have the following nice result.

THEOREM 1.1.1: *Let $S = \{(a_1\ a_2\ a_3 \mid a_4\ a_5 \mid a_6\ a_7\ a_8\ a_9 \mid \ldots \mid a_{n-1}, a_n) \mid a_i \in R; 1 \le i \le n\}$ be the collection of all super row vectors with same type of partition, S is a group under addition. Infact S is an abelian group of infinite order under addition.*

The proof is direct and hence left as an exercise to the reader.

If the field of reals R in Theorem 1.1.1 is replaced by Q the field of rationals or Z the integers or by the modulo integers Z_n, $n < \infty$ still the conclusion of the theorem 1.1.1 is true. Further the same conclusion holds good if the partitions are changed. S contains only same type of partition. However in case of Z_n, S becomes a finite commutative group.

Example 1.1.17: Let $P = \{(a_1 \mid a_2\ a_3 \mid a_4\ a_5\ a_6 \mid a_7\ a_8\ a_9\ a_{10}) \mid a_i \in Z; 1 \le i \le 10\}$ be the group of super row vectors; P is a group of infinite order. Clearly P has subgroups.

For take $H = \{(a_1 \mid a_2\ a_3 \mid a_4\ a_5\ a_6 \mid a_7\ a_8\ a_9\ a_{10}) \mid a_i \in 5Z; 1 \le i \le 10\} \subseteq P$ is a subgroup of super row vectors of infinite order.

Example 1.1.18: Let $G = \{(a_1\ a_2 \mid a_3\ ) \mid a_i \in Q; 1 \le i \le 3\}$ be a group of super row vectors.

Also we have group of super row vectors of the form given by the following examples.

Example 1.1.19: Let

$$G = \left\{\left(\begin{array}{cc|cccc|c} a_1 & a_4 & a_7 & a_{10} & a_{13} & a_{16} & a_{19} \\ a_2 & a_5 & a_8 & a_{11} & a_{14} & a_{17} & a_{20} \\ a_3 & a_6 & a_9 & a_{12} & a_{15} & a_{18} & a_{21} \end{array}\right) \middle| a_i \in Q, 1 \le i \le 21 \right\}$$

be the group of super row vectors under super row vector addition, G is an infinite commutative group.

Example 1.1.20: Let

$$V = \left\{\left(\begin{array}{ccc|cc} a_1 & a_3 & a_5 & a_7 & a_9 \\ a_2 & a_4 & a_6 & a_8 & a_{10} \end{array}\right) \middle| a_i \in R, 1 \le i \le 10 \right\}$$

be a group of super row vectors under addition.

Now having seen examples of group super row vectors now we proceed onto give examples of group of super column vectors. The definition can be made in an analogous way and this task is left as an exercise to the reader.

Example 1.1.21: Let

$$M = \left\{ \left(\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ \hline a_4 \\ a_5 \\ \hline a_6 \\ a_7 \\ a_8 \\ a_9 \end{array} \right) \middle| \, a_i \in Z, 1 \le i \le 9 \right\}$$

be the collection of super column vector with the same type of partition.

Now we can add any two elements in M and the sum is also in M.

For take

$$x = \left(\begin{array}{c} 3 \\ 2 \\ -1 \\ \hline 4 \\ -7 \\ \hline 8 \\ 0 \\ 5 \\ 2 \end{array} \right) \text{ and } y = \left(\begin{array}{c} 7 \\ 0 \\ 1 \\ \hline 2 \\ -1 \\ \hline 5 \\ 9 \\ 2 \\ -1 \end{array} \right)$$

in M. Now

$$x + y = \left(\begin{array}{c} 3 \\ 2 \\ -1 \\ \hline 4 \\ -7 \\ \hline 8 \\ 0 \\ 5 \\ 2 \end{array}\right) + \left(\begin{array}{c} 7 \\ 0 \\ 1 \\ \hline 2 \\ -1 \\ \hline 5 \\ 9 \\ 2 \\ -1 \end{array}\right) = \left(\begin{array}{c} 10 \\ 2 \\ 0 \\ \hline 6 \\ -8 \\ \hline 13 \\ 9 \\ 7 \\ 1 \end{array}\right)$$

is in M. Thus M is an additive abelian group of super column vectors of infinite order.

Clearly $\left(\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right)$ acts as the additive identity in M.

Example 1.1.22: Let

$$P = \left\{ \left(\begin{array}{c} a_1 \\ \hline a_2 \\ a_3 \\ \hline a_4 \\ a_5 \\ a_6 \\ \hline a_7 \\ a_8 \end{array}\right) \middle| \, a_i \in Q, 1 \leq i \leq 8 \right\}$$

be a group of super column vectors under addition.

Example 1.1.23: Let

$$M = \left\{ \left(\begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ \hline a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ \hline a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \end{array} \right) \middle| \, a_i \in R, 1 \le i \le 21 \right\}$$

be a group of super column vectors under addition of infinite order. Clearly M has subgroups.

For take

$$V = \left\{ \left(\begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ \hline a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ \hline a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \end{array} \right) \middle| \, a_i \in Q, 1 \le i \le 21 \right\} \subseteq M$$

is a subgroup of super column vectors of M.

Now we proceed onto give examples of groups formed out of super matrices with same type of partition.

Example 1.1.24: Let

$$M = \left\{ \left(\begin{array}{cc|ccc} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ \hline a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ \hline a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{26} & a_{27} & a_{28} & a_{29} & a_{30} \end{array} \right) \middle| \, a_i \in Q, 1 \le i \le 30 \right\}$$

be the collection of super matrices of the same type of partition.

Consider

$$x = \left[\begin{array}{cc|ccc} 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 0 & 1 & 2 \\ \hline 1 & 0 & 2 & 0 & 1 \\ \hline 3 & 0 & 1 & 2 & 0 \\ 1 & -4 & 2 & 0 & 1 \\ 2 & 0 & 0 & 1 & 2 \end{array}\right]$$

and

$$y = \left[\begin{array}{cc|ccc} 3 & 0 & 1 & 0 & 2 \\ 8 & 1 & 3 & 6 & 0 \\ \hline 7 & 6 & 1 & 6 & 2 \\ \hline 1 & 2 & 5 & 1 & 0 \\ 3 & 4 & 7 & 0 & 2 \\ 5 & 6 & 0 & 1 & -2 \end{array}\right]$$

be two super matrices in M.

$$x + y = \left[\begin{array}{cc|ccc} 3 & 1 & 3 & 3 & 6 \\ 13 & 7 & 3 & 7 & 2 \\ \hline 8 & 6 & 3 & 6 & 3 \\ \hline 4 & 2 & 6 & 3 & 0 \\ 4 & 0 & 9 & 0 & 3 \\ 7 & 6 & 0 & 2 & 0 \end{array}\right]$$

is in M and the sum is also of the same type.

Thus M is a commutative group of super matrices of infinite order.

Example 1.1.25: Let

$$P = \left\{ \left(\begin{array}{cc|c|cc} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ \hline a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ \hline a_{26} & a_{27} & a_{28} & a_{29} & a_{30} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{36} & a_{37} & a_{38} & a_{39} & a_{40} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{array} \right) \middle| \, a_i \in Z, 1 \leq i \leq 45 \right\}$$

be a group of super matrices of infinite order under addition. Consider the subgroups H = {A ∈ P | entries of A are from 3Z} ⊆ P; H is a subgroup of super matrices of infinite order. Infact P has infinitely many subgroups of super matrices.

Consider

$$W = \left\{ \left(\begin{array}{cc|c|cc} 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ \hline a_4 & a_5 & 0 & a_8 & a_9 \\ a_6 & a_7 & 0 & a_{10} & a_{11} \\ \hline 0 & 0 & a_{12} & 0 & 0 \\ 0 & 0 & a_{13} & 0 & 0 \\ 0 & 0 & a_{14} & 0 & 0 \\ 0 & 0 & a_{15} & 0 & 0 \end{array} \right) \middle| \, a_i \in 3Z, 1 \leq i \leq 15 \right\}$$

⊆ P is a subgroup of super matrices of infinite order under addition of P. Now we cannot make them as groups under multiplication for product of two super matrices do not in general turn out to be a super matrix of the desired form.

Next we leave for the reader to refer [47], for products of super matrices.

1.2 Refined Labels and Ordinary Labels and Their Properties

In this section we recall the notion of refined labels and ordinary labels and discuss the properties associated with them. Let $L_1, L_2, \ldots, L_m$ be labels where $m \geq 1$ is an integer.

Extend this set of labels where $m \geq 1$ with a minimum label L_0 and maximum label L_{m+1}. In case the labels are equidistant i.e., the qualitative labels is the same, we get an exact qualitative result, and the qualitative basic belief assignment (bba) is considered normalized if the sum of all its qualitative masses is equal to $L_{max} = L_{m+1}$ [48].

We consider a relation of order defined on these labels which can be "smaller" " less in quality" "lower" etc. $L_1 < L_2 < \ldots < L_m$. Connecting them to the classical interval [0, 1] we have $0 \equiv L_1 < L_2 < \ldots < L_i < \ldots < L_m < L_{m+1} \equiv 1$ and

$$L_i = \frac{i}{m+1}$$

for $i \in \{0, 1, 2, \ldots, m, m + 1\}$. [48] The set of labels $\tilde{L} \cong \{L_0, L_1, \ldots, L_m, L_{m+1}\}$ whose indices are positive integers between 0 and m+1, call the set 1-tuple labels. The labels may be equidistact or non equidistant [ ] call them as ordinary labels.

Now we say a set of labels $\{L_1, \ldots, L_m\}$ may not be totally orderable but only partially orderable. In such cases we have at least some $L_i < L_j$, $i \neq j$; $1 \leq i, j \leq m$, with L_0 the minimum element and L_{m+1} the maximum element. We call $\{L_0, L_1, \ldots, L_m, L_{m+1}\}$ a partially ordered set with $L_0 = 0$ and $L_{m+1} = 1$. Then we can define $\min\{L_i, L_j\} = L_i \cap L_j$ and $\max\{L_i, L_j\} = L_i \cup L_j$ where $L_i \cap L_j = L_k$ or 0 where $L_i > L_k$ and $L_j > L_k$ and $L_i \cup L_j = L_t$ or 1 where $L_t > L_i$ and $L_t > L_j$; $0 \leq k, t, i, j \leq m + 1$.

Thus $\{0 = L_0, L_1, \ldots, L_m, L_{m+1} = 1, \cup, \cap$ (min or max)$\}$ is a lattice.

When the ordinary labels are not totally ordered but partially ordered then the set of labels is a lattice. It can also happen that none of the attributes are "comparable" like in study

of fuzzy models only "related" in such cases they cannot be ordered in such case we can use the modulo integers models for the set of modulo integers Z_{m+1} ($m+1 < \infty$) ($n = m+1$) is unorderable.

Now [48] have defined refined labels we proceed onto describe a few of them only to make this book a self contained one. For more please refer [48]. The authors in [48] theoretically extend the set of labels L to the left and right sides of the interval [0, 1] towards $-\infty$ and respectively to ∞.

So define

$$L_Z \triangleq \left\{ \frac{j}{m+1} \middle/ j \in Z \right\}$$

where Z is the set of all positive and negative integers zero included.

Thus $L_z = \{.., L_{-j}, \ldots, L_{-1}, L_0, L_1, \ldots, L_j, \ldots\} = \{L_j / j \in Z\}$ that is the set of extended labels with positive and negative indexes.

Similarly one can define $L_Q \triangleq \{L_q / q \in Q\}$ as the set of labels whose indexes are fractions. L_Q is isomorphic with Q through the isomorphism $f_Q(L_q) = \dfrac{q}{m+1}$, for any $q \in Q$.

Even more general we can define

$$L_R \triangleq \left\{ \frac{r}{m+1} \middle/ r \in R \right\}$$

where R is the set of reals. L_R is isomorphic with R through the isomorphism $f_R(r) = \dfrac{r}{m+1}$ for any $r \in R$. The authors in [ ] prove that $\{L_R, +, \times\}$ is a field, where + is the vector addition of labels and $\times$ is the vector multiplication of labels which is called the DSm field of refined labels. Thus we introduce the decimal or refined labels i.e., labels whose index is decimal.

For instance $L_{3/10} = L_{0.3}$ and so on. For $[L_1, L_2]$, the middle of the label is $L_{1.5} = L_{3/2}$.

They have defined negative labels L_{-i} which is equal to $-L_i$.

$(L_R, +, \times, .)$ where '.' means scalar product is a commutative linear algebra over the field of real numbers R with unit element and each non null element in R is invertible with respect to multiplication of labels.

This is called the DSm field and Linear Algebra of Refined labels (FLARL for short).

Operators on FLARL is described below [48].

Let a, b, c $\in$ R and the labels

$$L_a = \frac{a}{m+1}, L_b = \frac{b}{m+1} \text{ and } L_c = \frac{c}{m+1}.$$

Let the scalars $\alpha, \beta \in$ R.

$$L_a + L_b = \frac{a}{m+1} + \frac{b}{m+1} = \frac{a+b}{m+1} = L_{a+b}.$$

$$L_a - L_b = \frac{a}{m+1} - \frac{b}{m+1} = \frac{a-b}{m+1}.$$

$$L_a \times L_b = L_{(ab)/m+1}$$

since

$$\frac{a}{m+1} \cdot \frac{b}{m+1} = \frac{ab/m+1}{m+1}.$$

For $\alpha \in$ R, we have

$$\alpha. L_a = L_a \alpha = L_{\alpha a}$$

since

$$\alpha.L_a = \alpha.\frac{a}{m+1} = \frac{\alpha a}{m+1}.$$

For

$$\alpha = -1; -1L_a = L_{-a} = -L_a.$$

Also

$$\frac{L_a}{\beta} = L_a \div \beta = \frac{1}{\beta} L_a = L_{a/\beta}.$$

$$(\beta \neq 0) \ (\because \ L_a = \frac{a}{m+1} \text{ and } \frac{1}{\beta} \times \frac{a}{m+1} = \frac{a}{\beta(m+1)}).$$

$$L_a \div L_b = L_{(a/b)(m+1)}$$

since

$$\frac{a}{m+1} \div \frac{b}{m+1} = \frac{a}{b} = \frac{(a/b)m+1}{m+1}$$

$$= L_{\left(\frac{a}{b}\right)m+1}.$$

$$(L_a)^p = L_{a^p/(m+1)^{p-1}}$$

since

$$\left(\frac{a}{m+1}\right)^p = \frac{a^p/(m+1)^{p-1}}{m+1}$$

for all $p \in R$.

$$\sqrt[k]{L_a} = (L_a)^{1/k}$$

$$= \quad L_{a^{1/k}/(m+1)^{1/k-1}},$$

this is got by replacing p by 1/k in $(L_a)^p$. [ ]

Further $(L_R, +, \times)$ is isomorphic with the set of real numbers $(R, +, \times)$; it results that $(L_R, +, \times)$ is also a field, called the DSm field of refined labels.

The field isomorphism

$$f_R : L_R \to R; \ f_R(L_r) = \frac{r}{m+1}$$

such that

$$f_R (L_a + L_b) = f_R (L_a) + f_R (L_b)$$

since

$$f_R (L_a + L_b) = f_R (L_{a+b})$$

$$= \frac{a+b}{m+1}$$

so

$$f_R (L_a) + f_R (L_b) = \frac{a}{m+1} + \frac{b}{m+1} = \frac{a+b}{m+1}.$$

$$f_R (L_a \times L_b) = f_R (L_a).\ f_R (L_b)$$

since

$$f_R (L_a \times L_b) = f_R (L_{(ab)/(m+1)})$$

$$= \frac{ab}{(m+1)^2}$$

and

$$f_R (L_a) \times f_R (L_b) = \frac{a}{m+1} . \frac{b}{m+1}$$

$$= \frac{a.b}{(m+1)^2}.$$

(L_R, +, .) is a vector space of refined labels over R.

It is pertinent to mention here that (L_R, +, .) is also a vector space over L_R as L_R is a field. So we make a deviation and since $L_R \cong R$, we need not in general distinguish the situation for we will treat both (L_R, +, .) a vector space over R or (L_R, +, .) a vector space over L_R as identical or one and the same as $L_R \cong R$.

Now we just recall some more operations [48].

Thus (L_R, +, $\times$, .) is a linear algebra of refined labels over the field R of real numbers called DSm linear algebra of refined labels which is commutative.

It is easily verified that multiplication in L_R is associative and multiplication is distributive with respect to addition.

L_{m+1} acts as the unitary element with respect to multiplication.

$$
\begin{aligned}
L_a . L_{m+1} &= L_{a.m+1} \\
&= L_{m+1.a} \\
&= L_{m+1}.L_a \\
&= L_{a(m+1)/m+1} \\
&= L_a.
\end{aligned}
$$

All $L_a \neq L_0$ are invertible,

$$(L_a)^{-1} = L_{(m+1)^2/a} = \frac{1}{L_a}.$$

$$
\begin{aligned}
L_a . (L_a)^{-1} &= L_a . \ L_{(m+1)^2/a} \\
&= L_{(a(m+1)^2/a)/m+1} \\
&= L_{m+1}.
\end{aligned}
$$

Also we have for $\alpha \in R$ and $L_a \in L_R$;

$$
\begin{aligned}
L_a + \alpha &= \alpha + L_a \\
&= L_{a+\alpha\ (m+1)}
\end{aligned}
$$

since

$$
\begin{aligned}
\alpha + L_a &= \frac{\alpha(m+1)}{m+1} + L_a \\
&= L_{\alpha(m+1)} + L_a \\
&= L_{\alpha(m+1)+a} \\
&= L_a + L_{\alpha(m+1)} \\
&= L_{a+\alpha\ (m+1)}.
\end{aligned}
$$

$$
\begin{aligned}
L_a - \alpha &= L_a - \frac{\alpha(m+1)}{m+1} \\
&= L_a - L_{\alpha\ (m+1)} \\
&= L_{a - \alpha\ (m+1)}
\end{aligned}
$$

and

$\alpha - L_a \quad = \quad L_{\alpha(m+1) - a}$.

Further

$$\frac{L_a}{\alpha} = L_a \times \frac{1}{\alpha} = L_{\frac{a}{\alpha}}$$

for a ≠ 0.

$$\alpha \div L_a = L_{\frac{\alpha(m+1)^2}{a}}$$

as

$$\begin{aligned} \alpha \div L_a &= \frac{\alpha(m+1)}{m+1} \div L_a \\ &= L_{\alpha(m+1)} \div L_a \\ &= L_{(\alpha(m+1)/a)\cdot m+1} \\ &= L_{\frac{\alpha}{a}(m+1)^2} \qquad [48]. \end{aligned}$$

It is important to make note of the following.

If R is replaced by Q still we see $L_Q \cong Q$ and we can say $(L_Q, +, .)$ is a linear algebra of refined labels over the field Q.

Further $(L_R, +, .)$ is also a linear algebra of refined labels over Q.

However $(L_Q, +, .)$ is not a linear algebra of refined labels over R. For more please refer [48].

Chapter Two

SUPERMATRICES OF REFINED LABELS

In this chapter we for the first time introduce the notion of super matrices whose entries are from L_R that is super matrices of refined labels and indicate a few of the properties enjoyed by them.

DEFINITION 2.1: *Let X = ($L_{a_1}, L_{a_2}, \ldots, L_{a_n}$) be a row matrix of refined labels with entries $L_{a_i} \in L_R$; $1 \leq i \leq n$. If X is partitioned in between the columns we get the super row matrix of refined labels.*

Example 2.1: Let X = $\left(L_{a_1} \;\; L_{a_2} \,\middle|\, L_{a_3} \;\; L_{a_4} \;\; L_{a_5} \;\; L_{a_6} \,\middle|\, L_{a_7}\right)$; $a_i \in L_R$; $1 \leq i \leq 7$, be a super row matrix (vector) of refined labels.

Example 2.2: Let

$$Y = \left(L_{b_1} \;\; L_{b_2} \;\; L_{b_3} \,\middle|\, L_{b_4} \,\middle|\, L_{b_5} \;\; L_{b_6} \;\; L_{b_7} \;\; L_{b_8} \,\middle|\, L_{b_9} \;\; L_{b_{10}}\right);$$

$L_{b_i} \in L_R$, $1 \leq i \leq 10$, be a super row matrix of refined labels.

Now for matrix of refined labels refer [48]. We can just say if we take a row matrix and replace the entries by the refined

labels in L_R and partition, the row matrix with refined labels then we get the super row matrix (vector) of refined labels.

Now we see that two $1 \times n$ super row matrices (vectors) are said to be of same type or similar partition or similar super row vectors if they are partitioned identically. For instance if x = $\left(L_{a_1} \left| L_{a_2} \;\; L_{a_3} \right| \ldots \left| L_{a_n} \right.\right)$ and y = $\left(L_{b_1} \left| L_{b_2} \;\; L_{b_3} \right| \ldots \left| L_{b_n} \right.\right)$ are two super row vectors of refined labels then x and y are similar or same type super row vectors.

Suppose z = $\left(L_{c_1} \;\; L_{c_2} \;\; L_{c_3} \left| L_{c_4} \;\; \ldots \right| L_{c_{n-1}} \;\; L_{c_n}\right)$ then z and x are not similar super row vectors as z enjoys a different partition from x and y.

Thus we see we can add two super row vectors of refined labels if and only if they hold the same type of partition. For instance if

$$x = \left(L_{a_1} \;\; L_{a_2} \left| L_{a_3} \right| L_{a_4} \;\; L_{a_5} \;\; L_{a_6}\right)$$

and

$$y = \left(L_{b_1} \;\; L_{b_2} \left| L_{b_3} \right| L_{b_4} \;\; L_{b_5} \;\; L_{b_6}\right)$$

where $L_{a_i}, L_{b_i} \in L_R$ we can add x and y as both x and y enjoy the same type of partition and both of them are of natural order 1×6.

$$x + y = \left(L_{a_1} \;\; L_{a_2} \left| L_{a_3} \right| L_{a_4} \;\; L_{a_5} \;\; L_{a_6}\right) + \left(L_{b_1} \;\; L_{b_2} \left| L_{b_3} \right| L_{b_4} \;\; L_{b_5} \;\; L_{b_6}\right)$$

$$= \left(L_{a_1} + L_{b_1} \;\; L_{a_2} + L_{b_2} \left| L_{a_3} + L_{b_3} \right| L_{a_4} + L_{b_4} \;\; L_{a_5} + L_{b_5} \;\; L_{a_6} + L_{b_6}\right)$$

$$= \left(L_{a_1+b_1} \;\; L_{a_2+b_2} \left| L_{a_3+b_3} \right| L_{a_4+b_4} \;\; L_{a_5+b_5} \;\; L_{a_6+b_6}\right).$$

We see x + y is of the same type as that of x and y. In view of this we have the following theorem.

THEOREM 2.1: *Let*

$$V = \left\{\left(L_{a_1} \;\; L_{a_2} \;\; L_{a_3} \left| L_{a_4} \;\; L_{a_5} \right| \ldots \left| L_{a_{n-1}} \;\; L_{a_n}\right.\right) \Big| L_{a_i} \in L_R; 1 \le i \le n\right\}$$

be the collection of all super row vectors of refined labels of same type. Then V is an abelian group under addition.

Proof is direct and hence is left as an exercise to the reader.

Now if

$$X = \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ \vdots \\ \hline L_{a_n} \end{array}\right]$$

denotes the super matrix column vector (matrix) of refined labels if

(i) Entries of X are elements from L_R.

(ii) X is a super matrix.

Thus if a usual (simple) row matrix is partitioned and the entries are taken from the field of refined labels then X is defined as the super column matrix (vector) of refined labels.
We will illustrate this situation by an example.

Example 2.3: Let

$$X = \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ L_{a_4} \\ L_{a_5} \\ \hline L_{a_6} \\ L_{a_7} \end{array}\right]$$

where $L_{a_i} \in L_R$; $1 \leq i \leq 7$ be the super column vector of refined labels.

Example 2.4: Consider the super column vector of refined labels given by

$$P = \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ \hline L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \\ L_{a_9} \\ L_{a_{10}} \end{array}\right]$$

where $L_{a_i} \in L_R$; $1 \leq i \leq 10$.

Example 2.5: Let

$$M = \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ \hline L_{a_3} \\ \hline L_{a_4} \end{array}\right]$$

where $L_{a_i} \in L_R$; $1 \leq i \leq 4$ is a super column vector of refined labels.

Now we can say as in case of super row vectors of refined labels, two super column vectors of refined labels are similar or of same type is defined in an analogous way.

We will illustrate this situation by some examples.

Consider

$$A = \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ L_{a_4} \\ \hline L_{a_5} \\ L_{a_6} \end{array}\right] \text{ and } B = \left[\begin{array}{c} L_{b_1} \\ \hline L_{b_2} \\ L_{b_3} \\ L_{b_4} \\ \hline L_{b_5} \\ L_{b_6} \end{array}\right]$$

be two super column vectors of same type of similar refined labels; $L_{a_i}, L_{b_j} \in L_R$; $1 \leq i, j \leq 6$. Take

$$P = \begin{bmatrix} L_{b_1} \\ \hline L_{b_2} \\ \hline L_{b_3} \\ \hline L_{b_4} \\ L_{b_5} \\ L_{b_6} \end{bmatrix}$$

be a super column vector of refined labels, we see P is not equivalent or similar or of same type as A and B.

Now having understood the concept of similar type of super column vectors we now proceed onto define addition. In the first place we have to mention that two super column vectors of refined labels are compatible under addition only if;

(1) Both the super column vectors of refined labels A and B must be of same natural order say $n \times 1$.
(2) Both A and B must be having the same type of partition.

Now we will illustrate this situation by some examples.

Let

$$A = \begin{bmatrix} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ L_{a_4} \\ \hline L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ \hline L_{a_8} \end{bmatrix} \text{ and } B = \begin{bmatrix} L_{b_1} \\ L_{b_2} \\ L_{b_3} \\ L_{b_4} \\ \hline L_{b_5} \\ L_{b_6} \\ L_{b_7} \\ \hline L_{b_8} \end{bmatrix}$$

be two super column vectors of refined labels over L_R; that is $L_{a_i}, L_{b_j} \in L_R$; $1 \leq i, j \leq 8$.

Clearly A and B are of same type super column vectors of refined labels.

Consider

$$M = \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ \hline L_{a_6} \\ L_{a_7} \\ \hline L_{a_8} \end{array}\right]$$

the super column vector of refined labels is not of same type A and B. Clearly all A, B and M are of same natural order 8×1 but are not of same type as M enjoys a different partition from A and B.

In view of this we have the following theorem.

THEOREM 2.2: *Let*

$$K = \left\{ \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ \hline \vdots \\ \hline L_{a_{n-2}} \\ L_{a_{n-1}} \\ \hline L_{a_n} \end{array}\right] \middle| \; L_{a_i} \in L_R; 1 \leq i \leq n \right\}$$

be a collection of super column vectors of refined labels of same type with entries from L_R. K is a group under addition of infinite order.

The proof is direct and simple hence left as an exercise to the reader.

Let

$$A = \begin{bmatrix} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ \hline L_{a_8} \\ L_{a_9} \end{bmatrix} \text{ and } B = \begin{bmatrix} L_{b_1} \\ \hline L_{b_2} \\ L_{b_3} \\ \hline L_{b_4} \\ L_{b_5} \\ L_{b_6} \\ L_{b_7} \\ \hline L_{b_8} \\ L_{b_9} \end{bmatrix}$$

where $L_{a_i}, L_{b_j} \in L_R$; $1 \le i, j \le 9$ and

$$A + B = \begin{bmatrix} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ \hline L_{a_8} \\ L_{a_9} \end{bmatrix} + \begin{bmatrix} L_{b_1} \\ \hline L_{b_2} \\ L_{b_3} \\ \hline L_{b_4} \\ L_{b_5} \\ L_{b_6} \\ L_{b_7} \\ \hline L_{b_8} \\ L_{b_9} \end{bmatrix} = \begin{bmatrix} L_{a_1} + L_{b_1} \\ \hline L_{a_2} + L_{b_2} \\ L_{a_3} + L_{b_3} \\ \hline L_{a_4} + L_{b_4} \\ L_{a_5} + L_{b_5} \\ L_{a_6} + L_{b_6} \\ L_{a_7} + L_{b_7} \\ \hline L_{a_8} + L_{b_8} \\ L_{a_9} + L_{b_9} \end{bmatrix} = \begin{bmatrix} L_{a_1+b_1} \\ \hline L_{a_2+b_2} \\ L_{a_3+b_3} \\ \hline L_{a_4+b_4} \\ L_{a_5+b_5} \\ L_{a_6+b_6} \\ L_{a_7+b_7} \\ \hline L_{a_8+b_8} \\ L_{a_9+b_9} \end{bmatrix}.$$

It is easily seen A + B is again of same type as that of A and B.

Now the super row vectors of refined labels and super column vectors of refined labels studied are simple super row vectors and simple column vectors of refined labels. We can

extend the results in case of super column vectors (matrices) and super row vectors (matrices) of refined labels which is not simple.

We give just examples of them.

Example 2.6: Let

$$V = \left[\begin{array}{ccc|cc|c} L_{a_1} & L_{a_4} & L_{a_7} & L_{a_{10}} & L_{a_{11}} & L_{a_{16}} \\ L_{a_2} & L_{a_5} & L_{a_8} & L_{a_{12}} & L_{a_{13}} & L_{a_{17}} \\ L_{a_3} & L_{a_6} & L_{a_9} & L_{a_{14}} & L_{a_{15}} & L_{a_{18}} \end{array}\right]$$

where $L_{a_i} \in L_R$; $1 \leq i \leq 18$ be a super row vector of refined labels.

Example 2.7: Let

$$M = \left[\begin{array}{cccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ \hline L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \\ L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \\ L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ \hline L_{a_{25}} & L_{a_{26}} & L_{a_{27}} & L_{a_{28}} \\ L_{a_{29}} & L_{a_{30}} & L_{a_{31}} & L_{a_{32}} \\ L_{a_{33}} & L_{a_{34}} & L_{a_{35}} & L_{a_{36}} \\ L_{a_{37}} & L_{a_{38}} & L_{a_{39}} & L_{a_{40}} \end{array}\right]$$

where $L_{a_i} \in L_R$; $1 \leq i \leq 40$ be a super column vector of refined labels.

When we have $m \times n$ matrix whose entries are refined labels we can partition them in many ways while by partition we get distinct super column vectors (super row vectors); so the study of super matrices gives us more and more matrices.

For instance take

$$V = \begin{bmatrix} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ L_{a_8} \end{bmatrix}$$

where $L_{a_i} \in L_R$; $1 \le i \le 8$. Now how many super matrices of refined labels can be built using V.

$$V_1 = \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ \hline L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ \hline L_{a_6} \\ \hline L_{a_7} \\ \hline L_{a_8} \end{array}\right], V_2 = \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ \hline L_{a_6} \\ \hline L_{a_7} \\ \hline L_{a_8} \end{array}\right], V_3 = \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ \hline L_{a_6} \\ \hline L_{a_7} \\ \hline L_{a_8} \end{array}\right], V_4 = \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ \hline L_{a_3} \\ L_{a_4} \\ \hline L_{a_5} \\ \hline L_{a_6} \\ \hline L_{a_7} \\ \hline L_{a_8} \end{array}\right],$$

$$V_5 = \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ \hline L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ \hline L_{a_6} \\ \hline L_{a_7} \\ \hline L_{a_8} \end{array}\right], V_6 = \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ \hline L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ \hline L_{a_8} \end{array}\right], V_7 = \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ \hline L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ \hline L_{a_6} \\ L_{a_7} \\ \hline L_{a_8} \end{array}\right], V_8 = \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ \hline L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ \hline L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \end{array}\right],$$

$$V_9 = \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ \hline L_{a_6} \\ \hline L_{a_7} \\ \hline L_{a_8} \end{array}\right], V_{10} = \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ \hline L_{a_6} \\ \hline L_{a_7} \\ \hline L_{a_8} \end{array}\right], V_{11} = \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ \hline L_{a_8} \end{array}\right], V_{12} = \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ \hline L_{a_6} \\ L_{a_7} \\ \hline L_{a_8} \end{array}\right],$$

$$V_{13} = \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ \hline L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \end{array}\right], V_{14} = \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ L_{a_4} \\ \hline L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \end{array}\right], V_{15} = \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ \hline L_{a_6} \\ L_{a_7} \\ \hline L_{a_8} \end{array}\right], V_{16} = \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \end{array}\right],$$

$$V_{17} = \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ \hline L_{a_6} \\ L_{a_7} \\ L_{a_8} \end{array}\right], V_{18} = \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ L_{a_4} \\ \hline L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \end{array}\right], V_{19} = \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \end{array}\right]$$

and so on.

Thus using a single ordinary column matrix of refined labels we obtain many super column vectors of refined labels. So we can for each type of partition on V get a group under addition. This will be exhibited from the following:

Consider

$$X = \begin{bmatrix} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ L_{a_4} \end{bmatrix}$$

be a column vector of refined labels.

$$X_1 = \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ L_{a_4} \end{array}\right], X_2 = \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ L_{a_4} \end{array}\right], X_3 = \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \end{array}\right], X_4 = \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ \hline L_{a_3} \\ L_{a_4} \end{array}\right],$$

$$X_5 = \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \end{array}\right], X_6 = \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ \hline L_{a_4} \end{array}\right] \text{ and } X_7 = \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ \hline L_{a_3} \\ \hline L_{a_4} \end{array}\right].$$

Thus we have seven super column vectors of refined labels for a given column vector refined label X.

Now if

$$T_1 = \left\{ X_1 = \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ L_{a_4} \end{array}\right] \middle| L_{a_i} \in L_R ; 1 \le i \le 4 \right\}$$

be the collection of all super column vectors of refined labels. Then T_1 is a group under addition of super column vectors of refined labels.

Likewise

$$T_2 = \left\{ X_2 = \begin{bmatrix} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ L_{a_4} \end{bmatrix} \middle| L_{a_i} \in L_R; 1 \le i \le 4 \right\}$$

is a super column vector group of refined labels.

Thus $T_i = \left\{ X_i \middle| L_{a_i} \in L_R; 1 \le i \le 4 \right\}$ is a super column vector of refined labels i=1,2, 3, …, 7. Hence we get seven distinct groups of super column vectors of refined labels.

Consider $Y = \left(L_{a_1} \; L_{a_2} \; L_{a_3} \; L_{a_4} \; L_{a_5} \right)$ be a super rows vector (matrix) of refined labels.

$$Y_1 = \left(L_{a_1} \middle| L_{a_2} \; L_{a_3} \; L_{a_4} \; L_{a_5} \right),\ Y_2 = \left(L_{a_1} \; L_{a_2} \middle| L_{a_3} \; L_{a_4} \; L_{a_5} \right)$$

$$Y_3 = \left(L_{a_1} \; L_{a_2} \; L_{a_3} \middle| L_{a_4} \; L_{a_5} \right),\ Y_4 = \left(L_{a_1} \; L_{a_2} \; L_{a_3} \; L_{a_4} \middle| L_{a_5} \right),$$

$$Y_5 = \left(L_{a_1} \middle| L_{a_2} \middle| L_{a_3} \; L_{a_4} \; L_{a_5} \right),\ Y_6 = \left(L_{a_1} \middle| L_{a_2} \; L_{a_3} \middle| L_{a_4} \; L_{a_5} \right),$$

$$Y_7 = \left(L_{a_1} \middle| L_{a_2} \; L_{a_3} \; L_{a_4} \middle| L_{a_5} \right),\ Y_8 = \left(L_{a_1} \; L_{a_2} \middle| L_{a_3} \middle| L_{a_4} \; L_{a_5} \right),$$

$$Y_9 = \left(L_{a_1} \; L_{a_2} \middle| L_{a_3} \; L_{a_4} \middle| L_{a_5} \right),\ Y_{10} = \left(L_{a_1} \; L_{a_2} \; L_{a_3} \middle| L_{a_4} \middle| L_{a_5} \right),$$

$$Y_{11} = \left(L_{a_1} \middle| L_{a_2} \middle| L_{a_3} \middle| L_{a_4} \; L_{a_5} \right),\ Y_{12} = \left(L_{a_1} \middle| L_{a_2} \middle| L_{a_3} \; L_{a_4} \middle| L_{a_5} \right),$$

$$Y_{13} = \left(L_{a_1} \middle| L_{a_2} \; L_{a_3} \middle| L_{a_4} \middle| L_{a_5} \right),\ Y_{14} = \left(L_{a_1} \; L_{a_2} \middle| L_{a_3} \middle| L_{a_4} \middle| L_{a_5} \right)$$

$$\text{and } Y_{15} = \left(L_{a_1} \middle| L_{a_2} \; L_{a_3} \middle| L_{a_4} \middle| L_{a_5} \right).$$

Thus we have 15 super row vectors of refined labels built using the row vector Y of refined labels. Hence we have 15 groups associated with the single row vector

$$X = \left(L_{a_1} \; L_{a_2} \; L_{a_3} \; L_{a_4} \; L_{a_5} \right)$$

of refined labels.

This is the advantage of using super row vectors in the place of row vectors of refined labels.

Consider the matrix

$$P = \begin{bmatrix} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{bmatrix}$$

of refined labels. How many super matrices of refined labels can be got from P.

$$P_1 = \left[\begin{array}{c|cc} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right], \; P_2 = \left[\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right],$$

$$P_3 = \left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right], \; P_4 = \left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right],$$

$$P_5 = \left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right], \; P_6 = \left[\begin{array}{c|c|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right],$$

$$P_7 = \left[\begin{array}{c|cc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right], \; P_8 = \left[\begin{array}{c|cc} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right],$$

$$P_9 = \left[\begin{array}{c|cc} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right], \; P_{10} = \left[\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right],$$

$$P_{11} = \left[\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right] \quad P_{12} = \left[\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right],$$

$$P_{13} = \left[\begin{array}{c|c|c} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right], P_{14} = \left[\begin{array}{c|c|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right],$$

$$P_{15} = \left[\begin{array}{c|c|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right], P_{16} = \left[\begin{array}{c|cc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right],$$

$$P_{17} = \left[\begin{array}{c|cc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right], P_{18} = \left[\begin{array}{c|cc} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right],$$

$$P_{19} = \left[\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right], P_{20} = \left[\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right],$$

$$P_{21} = \left[\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right], P_{22} = \left[\begin{array}{c|c|c} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right],$$

$$P_{23} = \left[\begin{array}{c|c|c} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right], \; P_{24} = \left[\begin{array}{c|c|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right],$$

$$P_{25} = \left[\begin{array}{c|cc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right], \; P_{26} = \left[\begin{array}{c|c|c} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right]$$

and

$$P_{27} = \left[\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right].$$

Thus the matrix P of refined labels gives us 27 super matrix of refined labels their by we can have 27 groups of super matrices of refined labels using $P_1, P_2, \ldots, P_{27}$. Now suppose we study the super matrix of refined labels using $L_{R^+\cup\{0\}} = \{L_r \mid r \in R^+ \cup \{0\}\}$ then also we can get the super matrix of refined labels. In this case any super matrix of refined labels collection of a particular type may not be a group but only a semigroup under addition. Let

$$X = \left\{\left(L_{a_1} \,\middle|\, L_{a_2} \;\; L_{a_3} \;\; L_{a_4} \,\middle|\, L_{a_5} \;\; L_{a_6}\right) \,\middle|\, L_{a_i} \in L_{R^+\cup\{0\}}\right\}$$

be a collection of super row vector of refined labels with entries from $L_{R^+\cup\{0\}}$. Clearly X is only a semigroup as no element has inverse.

Thus we have the following theorem.

THEOREM 2.3: *Let*

$$X = \left\{\left(L_{a_1} \;\; L_{a_2} \mid L_{a_3} \mid \ldots \mid L_{a_{n-1}} \;\; L_{a_n}\right) \,\middle|\, L_{a_i} \in L_{R^+\cup\{0\}}; 1 \leq i \leq n\right\}$$

be a collection of super row vectors of refined labels. X is a commutative semigroup with respect to addition.

The proof is direct and simple and hence left as an exercise to the reader.

Now consider

$$M = \left\{ \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ L_{a_6} \\ \hline \vdots \\ \hline L_{a_n} \end{array} \right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq n \right\}$$

be the super column vector of refined labels.

Now we say as in case of other labels that two super column vectors of refined labels x and y from $L_{R^+ \cup \{0\}}$ are equivalent or similar or of same type if both x and y have the same natural order and enjoy identical partition on it.

We can add only those two super column vector refined labels only when they have similar partition.

Consider

$$X = \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \end{array} \right] \text{ and } Y = \left[\begin{array}{c} L_{b_1} \\ L_{b_2} \\ L_{b_3} \\ \hline L_{b_4} \\ \hline L_{b_5} \\ L_{b_6} \\ \hline L_{b_7} \end{array} \right]$$

where $L_{a_i} \in L_{R^+ \cup \{0\}}$; $1 \leq i \leq 7$.

X+Y is defined and X+Y is again a super column vector of refined labels of same type. In view of this we have the following theorem.

THEOREM 2.4: *Let*

$$M = \left\{ \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ \hline \vdots \\ \hline L_{a_{n-2}} \\ L_{a_{n-1}} \\ L_{a_n} \end{array} \right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le n \right\}$$

be the collection of super column vectors of refined labels of same type with entries from $L_{R^+ \cup \{0\}}$. *M is a semigroup under addition.*

This proof is also direct and hence left as an exercise to the reader.

Now we show how two super column vectors of refined labels are added.

Let

$$X = \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ L_{a_4} \\ L_{a_5} \\ \hline L_{a_6} \\ L_{a_7} \\ \hline L_{a_8} \end{array} \right] \text{ and } Y = \left[\begin{array}{c} L_{b_1} \\ L_{b_2} \\ \hline L_{b_3} \\ L_{b_4} \\ L_{b_5} \\ \hline L_{b_6} \\ L_{b_7} \\ \hline L_{b_8} \end{array} \right]$$

be any two super column vectors of refined labels. Both X and Y are of same type and X+Y is also of the same type as X and Y. We see now as in case of usual super column vector of refined labels we have

$$X = \left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ \hline L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \end{array}\right]$$

is a super column vector of refined labels from the set $L_{R^+\cup\{0\}}$.

Likewise

$$Y = \left\{ \left[\begin{array}{c|ccc|cc} L_{a_1} & L_{a_5} & L_{a_9} & L_{a_{13}} & L_{a_{17}} & L_{a_{21}} \\ L_{a_2} & L_{a_6} & L_{a_{10}} & L_{a_{14}} & L_{a_{18}} & L_{a_{22}} \\ L_{a_3} & L_{a_7} & L_{a_{11}} & L_{a_{15}} & L_{a_{19}} & L_{a_{23}} \\ L_{a_4} & L_{a_8} & L_{a_{12}} & L_{a_{16}} & L_{a_{20}} & L_{a_{24}} \end{array}\right] \middle| L_{a_i} \in L_{R^+\cup\{0\}} \right\}$$

is a super row vector of refined labels and it is easily verified Y is not a simple super row vector of refined labels.

Now it is interesting to note $L_{R^+\cup\{0\}}$ = {set of all refined labels with positive value} is a semifield of refined labels or DSm semifield of refined labels and $L_{R^+\cup\{0\}} \cong R^+ \cup \{0\}$ [48].

Thus we see further

$$A = \left[\begin{array}{cc|c|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ \hline L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right]$$

is a super matrix of refined labels with entries from $L_{R^+\cup\{0\}}$.

Suppose

$$A = \left[\begin{array}{ccc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array}\right]$$

be a super matrix of refined labels with entries from $L_{R^+\cup\{0\}}$. Now if we consider super matrices of refined labels with entries from $L_{R^+\cup\{0\}}$ of same natural order and similar type of partition then addition can be defined.

We will just show how addition is defined when super matrices of refined labels are of same type.

$$\text{Let } A = \left[\begin{array}{cc|ccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ \hline L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \end{array}\right] \text{ and}$$

$$B = \left[\begin{array}{cc|ccc} L_{b_1} & L_{b_2} & L_{b_3} & L_{b_4} & L_{b_5} \\ L_{b_6} & L_{b_7} & L_{b_8} & L_{b_9} & L_{b_{10}} \\ \hline L_{b_{11}} & L_{b_{12}} & L_{b_{13}} & L_{b_{14}} & L_{b_{15}} \\ L_{b_{16}} & L_{b_{17}} & L_{b_{18}} & L_{b_{19}} & L_{b_{20}} \end{array}\right]$$

be any two super matrices of refined labels with entries from $L_{R^+\cup\{0\}}$; $1 \leq i \leq 20$.

$$A + B = \left[\begin{array}{cc|ccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ \hline L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \end{array}\right] +$$

$$\left[\begin{array}{cc|ccc}
L_{b_1} & L_{b_2} & L_{b_3} & L_{b_4} & L_{b_5} \\
L_{b_6} & L_{b_7} & L_{b_8} & L_{b_9} & L_{b_{10}} \\
\hline
L_{b_{11}} & L_{b_{12}} & L_{b_{13}} & L_{b_{14}} & L_{b_{15}} \\
L_{b_{16}} & L_{b_{17}} & L_{b_{18}} & L_{b_{19}} & L_{b_{20}}
\end{array}\right]$$

$$= \left[\begin{array}{cc|ccc}
L_{a_1}+L_{b_1} & L_{a_2}+L_{b_2} & L_{a_3}+L_{b_3} & L_{a_4}+L_{b_4} & L_{a_5}+L_{b_5} \\
L_{a_6}+L_{b_6} & L_{a_7}+L_{b_7} & L_{a_8}+L_{b_8} & L_{a_9}+L_{b_9} & L_{a_{10}}+L_{b_{10}} \\
\hline
L_{a_{11}}+L_{b_{11}} & L_{a_{12}}+L_{b_{12}} & L_{a_{13}}+L_{b_{13}} & L_{a_{14}}+L_{b_{14}} & L_{a_{15}}+L_{b_{15}} \\
L_{a_{16}}+L_{b_{16}} & L_{a_{17}}+L_{b_{17}} & L_{a_{18}}+L_{b_{18}} & L_{a_{19}}+L_{b_{19}} & L_{a_{20}}+L_{b_{20}}
\end{array}\right]$$

$$= \left[\begin{array}{cc|ccc}
L_{a_1+b_1} & L_{a_2+b_2} & L_{a_3+b_3} & L_{a_4+b_4} & L_{a_5+b_5} \\
L_{a_6+b_6} & L_{a_7+b_7} & L_{a_8+b_8} & L_{a_9+b_9} & L_{a_{10}+b_{10}} \\
\hline
L_{a_{11}+b_{11}} & L_{a_{12}+b_{12}} & L_{a_{13}+b_{13}} & L_{a_{14}+b_{14}} & L_{a_{15}+b_{15}} \\
L_{a_{16}+b_{16}} & L_{a_{17}+b_{17}} & L_{a_{18}+b_{18}} & L_{a_{19}+b_{19}} & L_{a_{20}+b_{20}}
\end{array}\right].$$

We see A+B, A and B are of same type. In view of this we have the following theorem.

THEOREM 2.5: *Let* $A = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle|\right.$ *a_i are submatrices of refined labels with entries from* $L_{R^+ \cup \{0\}}$*; $1 \leq i \leq 9$}. A is a semigroup under addition.*

Here A is a collection of all super matrices of refined labels where a_1, a_2 and a_3 are submatrices of refined labels with same number of rows. Also a_4, a_5 and a_6 are submatrices of refined labels with same number of rows. a_7, a_8 and a_9 are submatrices of refined labels with same number of rows.

Similarly a_1, a_4 and a_7 are submatrices of refined labels with same number of columns. a_2, a_5 and a_6 are submatrices of

refined labels with same number of columns. a_3, a_6 and a_9 are submatrices of refined labels with same number of columns. Thus A is a semigroup under addition.

Now $L_{Q^+ \cup \{0\}}$ gives the collection of all refined labels and $L_{Q^+ \cup \{0\}} \cong Q^+ \cup \{0\}$, infact $L_{Q^+ \cup \{0\}} \subseteq L_{R^+ \cup \{0\}}$.

Suppose we take the labels $0 = L_0 < L_1 < L_2 < \ldots < L_m = L_{m+1} = 1$, that the collection of ordinary labels; then with min or max or equivalently $\cap$ or $\cup$, these labels are semilattices or semigroups. That is $L = \{0 = L_0 < L_1 < L_2 < \ldots < L_m < L_{m+1} = 1\}$ is a semilattice. Infact when both $\cup$ and $\cap$ are defined L is a lattice called the chain lattice.

Let P = $\left\{\left(L_{a_1} \quad L_{a_2} \mid L_{a_3} \quad L_{a_4} \quad L_{a_5}\right) \middle| L_{a_i} \in L\right\}$ = $L_0 < L_1 < \ldots < L_m < L_{m+1} = 1\}$ be a super row vector of ordinary labels. We say two super row vector ordinary labels are similar or identical if they are of same natural order and hold same or identical partition on it. We call such super row vectors of ordinary labels as super row vectors of same type.

Let

$$X = \left(L_{a_1} \mid L_{a_2} \quad L_{a_3} \mid L_{a_4} \quad L_{a_5} \quad L_{a_6}\right) \text{ and}$$

$$Y = \left(L_{b_1} \mid L_{b_2} \quad L_{b_3} \mid L_{b_4} \quad L_{b_5} \quad L_{b_6}\right)$$

be two super row vectors of ordinary labels; $L_{a_i}, L_{b_j} \in L = \{0 = L_0 < L_1 < L_2 < \ldots < L_m < L_{m+1} = 1\}$, $1 \leq i, j \leq 6$. Now '+' cannot be defined. We define '$\cup$' which is max (L_{a_i}, L_{b_j}).

Thus $X \cup Y = \left(L_{a_1} \mid L_{a_2} \quad L_{a_3} \mid L_{a_4} \quad L_{a_5} \quad L_{a_6}\right) \cup$

$\left(L_{b_1} \mid L_{b_2} \quad L_{b_3} \mid L_{b_4} \quad L_{b_5} \quad L_{b_6}\right)$

$= \left(L_{a_1} \cup L_{b_1} \mid L_{a_2} \cup L_{b_2} \quad L_{a_3} \cup L_{b_3} \mid L_{a_4} \cup L_{b_4} \quad L_{a_5} \cup L_{b_5} \quad L_{a_6} \cup L_{b_6}\right)$

$= (\max\{L_{a_1}, L_{b_1}\} \mid \max \{L_{a_2}, L_{b_2}\}, \max \{L_{a_3}, L_{b_3}\} \mid \max \{L_{a_4}, L_{b_4}\} \max \{L_{a_5}, L_{b_5}\} \max \{L_{a_6}, L_{b_6}\})$

$= \left(L_{c_1} \mid L_{c_2} \quad L_{c_3} \mid L_{c_4} \quad L_{c_5} \quad L_{c_6}\right)$.

We see X, Y and X $\cup$ Y are same type of super row vectors of ordinary labels.

In an analogous way we can define X $\cap$ Y or min {X,Y}.

X $\cap$Y is again of the same type as X and Y.

Now we have the following interesting results.

THEOREM 2.6: *Let V =*

$$\left\{\left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \mid L_{a_4} \mid L_{a_5} \quad L_{a_6} \mid L_{a_7} \quad \ldots \mid \ldots \mid L_{a_{n-1}} \quad L_{a_n}\right) \middle| L_{a_i}\right.$$

$\in L= \{0 = L_0 < L_1 < \ldots < L_m < L_{m+1}= 1\}$; $1 \leq i \leq a_n\}$ be the collection of super row vectors of ordinary labels. V is a semigroup under '$\cup$'.

THEOREM 2.7: *Let V=*

$$\left\{\left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \mid L_{a_4} \mid L_{a_5} \quad L_{a_6} \mid L_{a_7} \quad \ldots \mid \ldots \mid L_{a_{n-1}} \quad L_{a_n}\right) \middle| L_{a_i}\right.$$

$\in L = \{0 = L_0 < L_1 < \ldots < L_m = L_{m+1} = 1\}$; $1 \leq i \leq a_n\}$ be the collection of super row vectors of ordinary labels. V is a semigroup under '$\cap$'.

The proof is left as an exercise to the reader.

In the theorem 2.7 if '$\cap$' is replaced by '$\cup$', V is a semigroup of super row matrices (vector) of ordinary labels.

Now consider

$$P = \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \end{array}\right]$$

where $L_{a_i} \in L = \{0 = L_0 < L_1$
$< . . . < L_m < L_{m+1}=1\}$, $1\leq i \leq 8$, P is a super column vector of ordinary labels. Now we cannot in general define $\cup$ or $\cap$ on any two super column vectors of ordinary labels even if they are of same natural order. Any two super column vectors of ordinary labels of same natural order can have $\cup$ or $\cap$ defined on them only when the both of them enjoy the same type of partition on it.

We will now give a few illustrations before we proceed on to suggest some related results.

Let

$$T = \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \\ \hline L_{a_9} \end{array}\right] \text{ and } S = \left[\begin{array}{c} L_{b_1} \\ \hline L_{b_2} \\ L_{b_3} \\ \hline L_{b_4} \\ L_{b_5} \\ L_{b_6} \\ \hline L_{b_7} \\ L_{b_8} \\ \hline L_{b_9} \end{array}\right]$$

be two super column vector matrices of ordinary labels where $L_{a_i}, L_{b_j} \in \{L; 0 = L_0 < L_1 < L_2 < ... < L_m < L_{m+1}\}$; $1 \leq i , j \leq 9$. Now we define min $\{T, S\} = T \cap S$ as follows min $\{T,S\}=$

$$\min\left\{\begin{bmatrix} \frac{L_{a_1}}{L_{a_2}} \\ \frac{L_{a_3}}{L_{a_4}} \\ L_{a_5} \\ \frac{L_{a_6}}{L_{a_7}} \\ \frac{L_{a_8}}{L_{a_9}} \end{bmatrix}, \begin{bmatrix} \frac{L_{b_1}}{L_{b_2}} \\ \frac{L_{b_3}}{L_{b_4}} \\ L_{b_5} \\ \frac{L_{b_6}}{L_{b_7}} \\ \frac{L_{b_8}}{L_{b_9}} \end{bmatrix}\right\} = \begin{bmatrix} \frac{\min\{L_{a_1}, L_{b_1}\}}{\min\{L_{a_2}, L_{b_2}\}} \\ \frac{\min\{L_{a_3}, L_{b_3}\}}{\min\{L_{a_4}, L_{b_4}\}} \\ \min\{L_{a_5}, L_{b_5}\} \\ \frac{\min\{L_{a_6}, L_{b_6}\}}{\min\{L_{a_7}, L_{b_7}\}} \\ \frac{\min\{L_{a_8}, L_{b_8}\}}{\min\{L_{a_9}, L_{b_9}\}} \end{bmatrix}$$

$$= T \cap S = \begin{bmatrix} \frac{L_{a_1}}{L_{a_2}} \\ \frac{L_{a_3}}{L_{a_4}} \\ L_{a_5} \\ \frac{L_{a_6}}{L_{a_7}} \\ \frac{L_{a_8}}{L_{a_9}} \end{bmatrix} \cap \begin{bmatrix} \frac{L_{b_1}}{L_{b_2}} \\ \frac{L_{b_3}}{L_{b_4}} \\ L_{b_5} \\ \frac{L_{b_6}}{L_{b_7}} \\ \frac{L_{b_8}}{L_{b_9}} \end{bmatrix} = \begin{bmatrix} \frac{L_{a_1} \cap L_{b_1}}{L_{a_2} \cap L_{b_2}} \\ \frac{L_{a_3} \cap L_{b_3}}{L_{a_4} \cap L_{b_4}} \\ L_{a_5} \cap L_{b_5} \\ \frac{L_{a_6} \cap L_{b_6}}{L_{a_7} \cap L_{b_7}} \\ \frac{L_{a_8} \cap L_{b_8}}{L_{a_9} \cap L_{b_9}} \end{bmatrix} = \begin{bmatrix} \frac{L_{a_1 \cap b_1}}{L_{a_2 \cap b_2}} \\ \frac{L_{a_3 \cap b_3}}{L_{a_4 \cap b_4}} \\ L_{a_5 \cap b_5} \\ \frac{L_{a_6 \cap b_6}}{L_{a_7 \cap b_7}} \\ \frac{L_{a_8 \cap b_8}}{L_{a_9 \cap b_9}} \end{bmatrix}$$

$$= \left[\begin{array}{c} L_{\min\{a_1,b_1\}} \\ \hline L_{\min\{a_2,b_2\}} \\ L_{\min\{a_3,b_3\}} \\ \hline L_{\min\{a_4,b_4\}} \\ L_{\min\{a_5,b_5\}} \\ L_{\min\{a_6,b_6\}} \\ \hline L_{\min\{a_7,b_7\}} \\ L_{\min\{a_8,b_8\}} \\ \hline L_{\min\{a_9,b_9\}} \end{array}\right].$$

We make a note of the following.

This min = $\cap$ (or max = $\cup$) are defined on super column vectors of ordinary labels only when they are of same natural order and enjoy the same type of partition.

THEOREM 2.8: *Let*

$$K = \left\{ \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ \hline \vdots \\ \hline L_{a_{n-1}} \\ L_{a_n} \end{array}\right] \middle| \, L_{a_i} \in L \right.$$

= {0 = L_0 < L_1 < L_2 < … < L_m < L_{m+1} = 1}} be the collection of all super column vectors of ordinary labels which enjoy the same form of partition.

(i) *{K, ∪} is a semigroup of finite order which is commutative (semilattice of finite order)*

(ii) *{K ,∩} is a semigroup of finite order which is commutative (semilattice of finite order)*

(iii) *{K ,∪, ∩} is a lattice of finite order.*

Now we study super matrices of ordinary labels.

Consider

$$H = \left[\begin{array}{c|c} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} \end{array}\right]$$

is a super matrix of ordinary labels.

Let

$$M = \left[\begin{array}{c|c} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} \end{array}\right]$$

again a super matrix of ordinary labels with entries from L= {0 = $L_0 < L_1 < L_2 < \ldots < L_m < L_{m+1} = 1$}.

Take

$$P = \left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} \end{array}\right],$$

P is again a super matrix of ordinary labels.

$$R = \left[\begin{array}{c|c} L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} \end{array}\right]$$

is a supermatrix of ordinary labels.

$$V = \left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} \end{array}\right] \text{ and } W = \left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} \end{array}\right]$$

are super matrices ordinary labels.

$$T = \left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} \end{array}\right] \text{ and } B = \left[\begin{array}{c|c} L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} \end{array}\right]$$

are super matrices of ordinary labels.

$$C = \left[\begin{array}{c|c} L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} \end{array}\right] \text{ and } D = \left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} \end{array}\right]$$

are super matrices of ordinary labels.

$$E = \left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} \end{array}\right] \text{ and } F = \left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} \end{array}\right]$$

are super matrices of ordinary labels.

$$G = \left[\begin{array}{c|c} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} \end{array}\right] \text{ and } L = \left[\begin{array}{c|c} L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} \end{array}\right]$$

are super matrices of ordinary labels.

Now

$$Q = \left[\begin{array}{c|c} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} \end{array}\right]$$

is again a super matrix of ordinary labels. Thus we have H , M , P , R , V , W , T , B , C , D , E , F , G , L and Q; 15 super matrices of ordinary labels of natural order 4×2.

Now we can also get 15 super matrix semigroups under '$\cup$' and 15 super matrix semigroup under $\cap$ of finite order.

Thus by converting an ordinary label matrices into super matrices makes one get several matrices out of a single matrix.

THEOREM 2.9: *Let A = {all m × n super matrices of ordinary labels of same type of partition}. A is a semigroup under $\cap$ (or semigroup under $\cup$) of finite order which is commutative.*

Now having seen examples and theorems about super matrices of ordinary labels, we in the chapter four proceed on to define linear algebra and vector spaces of super matrices. It is important to note that in general usual products are not defined among super matrices. In chapter three we define products and transpose of super label matrices. However even the collection of super matrices of same type are not closed in general under product.

Chapter Three

OPERATIONS ON SUPERMATRICES OF REFINED LABELS

In this chapter we proceed on to define transpose of a super matrix of refined labels (ordinary labels) and define product of super matrices of refined labels.

Let

$$A = \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ L_{a_7} \end{array}\right]$$

be a super column matrix of refined labels with entries from L_R. Clearly the transpose of A denoted by

$$A^t = \begin{bmatrix} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ L_{a_7} \end{bmatrix}^t = \left[L_{a_1} \quad L_{a_2} \mid L_{a_3} \mid L_{a_4} \quad L_{a_5} \quad L_{a_6} \quad L_{a_7}\right];$$

Thus we see the transpose of A is the super row vector (matrix) of refined labels. If we replace the refined labels by ordinary labels also the results holds good.

Consider

$$P = \left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \mid L_{a_4} \mid L_{a_5} \quad L_{a_6} \quad L_{a_7} \quad L_{a_8} \mid L_{a_9} \quad L_{a_{10}}\right)$$

be a super row vector (matrix) of refined labels.

Now $L_{a_i} \in L_R$; $1 \leq i \leq 9$. Now the transpose of P denoted by P^t is

$$\left(L_{a_1} \;\; L_{a_2} \;\; L_{a_3} \mid L_{a_4} \mid L_{a_5} \;\; L_{a_6} \;\; L_{a_7} \;\; L_{a_8} \mid L_{a_9} \;\; L_{a_{10}}\right)^t = \begin{bmatrix} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ L_{a_8} \\ \hline L_{a_9} \\ L_{a_{10}} \end{bmatrix};$$

we see P^t is a super column matrix (vector) of refined labels. However if we replace the refined labels by ordinary labels still the results hold good.

Let

$$A = \left[\begin{array}{ccc|cc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ \hline L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \\ L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} & L_{a_{25}} \end{array}\right]$$

be a super matrix of refined labels with entries from L_R; ie $L_{a_i} \in L_R$; $1 \leq i \leq 25$.

$$A^t = \left[\begin{array}{ccc|cc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ \hline L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \\ L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} & L_{a_{25}} \end{array}\right]^t$$

$$= \left[\begin{array}{cc|ccc} L_{a_1} & L_{a_6} & L_{a_{11}} & L_{a_{16}} & L_{a_{21}} \\ L_{a_2} & L_{a_7} & L_{a_{12}} & L_{a_{17}} & L_{a_{22}} \\ L_{a_3} & L_{a_8} & L_{a_{13}} & L_{a_{18}} & L_{a_{23}} \\ \hline L_{a_4} & L_{a_9} & L_{a_{14}} & L_{a_{19}} & L_{a_{24}} \\ L_{a_5} & L_{a_{10}} & L_{a_{15}} & L_{a_{20}} & L_{a_{25}} \end{array}\right]$$

is again a supermatrix of refined labels with entries from L_R; $L_{a_i} \in L_R$; $1 \leq i \leq 25$.

Let

$$P = \left[\begin{array}{cc|ccc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ \hline L_{a_{19}} & L_{a_{20}} & L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ L_{a_{25}} & L_{a_{26}} & L_{a_{27}} & L_{a_{28}} & L_{a_{29}} & L_{a_{30}} \\ L_{a_{31}} & L_{a_{32}} & L_{a_{33}} & L_{a_{34}} & L_{a_{35}} & L_{a_{36}} \end{array}\right]$$

be a super matrix of refined labels with entries from L_R; $L_{a_i} \in L_R$; $1 \leq i \leq 36$.

$$P^t = \left[\begin{array}{cc|ccc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ \hline L_{a_{19}} & L_{a_{20}} & L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ L_{a_{25}} & L_{a_{26}} & L_{a_{27}} & L_{a_{28}} & L_{a_{29}} & L_{a_{30}} \\ L_{a_{31}} & L_{a_{32}} & L_{a_{33}} & L_{a_{34}} & L_{a_{35}} & L_{a_{36}} \end{array}\right]^t$$

$$= \left[\begin{array}{c|cc|ccc} L_{a_1} & L_{a_7} & L_{a_{13}} & L_{a_{19}} & L_{a_{25}} & L_{a_{31}} \\ L_{a_2} & L_{a_8} & L_{a_{14}} & L_{a_{20}} & L_{a_{26}} & L_{a_{32}} \\ \hline L_{a_3} & L_{a_9} & L_{a_{15}} & L_{a_{21}} & L_{a_{27}} & L_{a_{33}} \\ L_{a_4} & L_{a_{10}} & L_{a_{16}} & L_{a_{22}} & L_{a_{28}} & L_{a_{34}} \\ L_{a_5} & L_{a_{11}} & L_{a_{17}} & L_{a_{23}} & L_{a_{29}} & L_{a_{35}} \\ \hline L_{a_6} & L_{a_{12}} & L_{a_{18}} & L_{a_{24}} & L_{a_{30}} & L_{a_{36}} \end{array}\right]$$

is again a supermatrix of refined labels with entries from L_R; $L_{a_i} \in L_R$; $1 \leq i \leq 36$.

Suppose

$$X = \left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ \hline L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \\ L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ L_{a_{25}} & L_{a_{26}} & L_{a_{27}} \end{array}\right]$$

be a super column vector (matrix); with entries from L_R ($L_{a_i} \in L_R$; $1 \leq i \leq 27$) .

The transpose of X denoted by

$$X^t = \left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ \hline L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \\ L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ L_{a_{25}} & L_{a_{26}} & L_{a_{27}} \end{array}\right]^t$$

$$= \left[\begin{array}{c|cc|ccc|ccc} L_{a_1} & L_{a_4} & L_{a_7} & L_{a_{10}} & L_{a_{13}} & L_{a_{16}} & L_{a_{19}} & L_{a_{22}} & L_{a_{25}} \\ L_{a_2} & L_{a_5} & L_{a_8} & L_{a_{11}} & L_{a_{14}} & L_{a_{17}} & L_{a_{20}} & L_{a_{23}} & L_{a_{26}} \\ L_{a_3} & L_{a_6} & L_{a_9} & L_{a_{12}} & L_{a_{15}} & L_{a_{18}} & L_{a_{21}} & L_{a_{24}} & L_{a_{27}} \end{array}\right]$$

is a super row vector (matrix) of refined labels with entries from L_R.

Let

$$M = \left[\begin{array}{cc|c|cccc} L_{a_1} & L_{a_7} & L_{a_{13}} & L_{a_{19}} & L_{a_{25}} & L_{a_{31}} & L_{a_{37}} \\ L_{a_2} & L_{a_8} & L_{a_{14}} & L_{a_{20}} & L_{a_{26}} & L_{a_{32}} & L_{a_{38}} \\ L_{a_3} & L_{a_9} & L_{a_{15}} & L_{a_{21}} & L_{a_{27}} & L_{a_{33}} & L_{a_{39}} \\ L_{a_4} & L_{a_{10}} & L_{a_{16}} & L_{a_{22}} & L_{a_{28}} & L_{a_{34}} & L_{a_{40}} \\ L_{a_5} & L_{a_{11}} & L_{a_{17}} & L_{a_{23}} & L_{a_{29}} & L_{a_{35}} & L_{a_{41}} \\ L_{a_6} & L_{a_{12}} & L_{a_{18}} & L_{a_{24}} & L_{a_{30}} & L_{a_{36}} & L_{a_{42}} \end{array}\right]$$

be a super row vector (matrix) of refined labels with entries from L_R.

$$M^t = \left[\begin{array}{cccccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ \hline L_{a_{19}} & L_{a_{20}} & L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ L_{a_{25}} & L_{a_{26}} & L_{a_{27}} & L_{a_{28}} & L_{a_{29}} & L_{a_{30}} \\ L_{a_{31}} & L_{a_{32}} & L_{a_{33}} & L_{a_{34}} & L_{a_{35}} & L_{a_{36}} \\ L_{a_{37}} & L_{a_{38}} & L_{a_{39}} & L_{a_{40}} & L_{a_{41}} & L_{a_{42}} \end{array}\right]$$

is the transpose of M which is a super column vector (matrix) of refined labels with entries from L_R.

Now we find the transpose of super matrix of refined labels which are not super row vectors or super column vectors.

Let

$$P = \left[\begin{array}{ccc|cc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ \hline L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \\ L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} & L_{a_{25}} \end{array}\right]$$

where $L_{a_i} \in L_R$; $1 \leq i \leq 25$ be a super matrix of refined labels.

$$P^t = \left[\begin{array}{ccc|cc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ \hline L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \\ L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} & L_{a_{25}} \end{array}\right]$$

is the transpose of P and is again a super matrix of refined labels.

Now we will show how product is defined in case of super matrix of refined labels. We just make a mention that if in the super matrix the refined labels are replaced by ordinary labels still the concept of transpose of super matrices remain the same

however there will be difference in the notion of product which will be discussed in the following:

Suppose $A = \left[L_{a_1} \quad L_{a_2} \mid L_{a_3} \quad L_{a_4} \quad L_{a_5}\right]$ be a super matrix (row vector) of refined labels then we find

$$A^t = \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ L_{a_4} \\ L_{a_5} \end{array}\right],$$

which is a super column vector of refined labels.

We find

$$AA^t = \left[L_{a_1} \quad L_{a_2} \mid L_{a_3} \quad L_{a_4} \quad L_{a_5}\right] \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ L_{a_4} \\ L_{a_5} \end{array}\right]$$

$$= \left[L_{a_1} \quad L_{a_2}\right] \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \end{array}\right] + \left[L_{a_3} \quad L_{a_4} \quad L_{a_5}\right] \left[\begin{array}{c} L_{a_3} \\ L_{a_4} \\ L_{a_5} \end{array}\right]$$

$$= \left[L_{a_1}.L_{a_1} + L_{a_2}.L_{a_2}\right] + \left[L_{a_3}.L_{a_3} + L_{a_4}.L_{a_4} + L_{a_5}.L_{a_5}\right]$$

$$= \left[L_{a_1^2/(m+1)} + L_{a_2^2/(m+1)}\right] + \left[L_{a_3^2/(m+1)} + L_{a_4^2/(m+1)} + L_{a_5^2/(m+1)}\right]$$

$$= \left[L_{a_1^2/(m+1)+a_2^2/(m+1)}\right] + \left[L_{a_3^2/(m+1)+a_4^2/(m+1)+a_5^2/(m+1)}\right]$$

$$= \left[L_{a_1^2+a_2^2/(m+1)}\right] + \left[L_{a_3^2+a_4^2+a_5^2/(m+1)}\right]$$

$$= \left[L_{a_1^2+a_2^2+a_3^2+a_4^2+a_5^2/(m+1)}\right].$$

Clearly $AA^t \in L_R$ but it is not a super row vector matrix. Now we see if we want to multiply a super row vector A with a

super column vector B of refined labels then we need the following.

(i) If A is of natural order $1 \times n$ then B must be of natural order $n \times 1$.

(ii) If A is partitioned between the columns a_i and a_{i+1} then B must be partitioned between the rows b_i and b_{i+1} of B this must be carried out for every $1 \leq i \leq n$.

We will illustrate this situation by some examples.

Let $A = \left[L_{a_1} \quad L_{a_2} \mid L_{a_3} \quad L_{a_4} \quad L_{a_5} \mid L_{a_6}\right]$ be a super row vector of natural order 1×6. Consider $B = \left[\begin{array}{c} L_{b_1} \\ L_{b_2} \\ \hline L_{b_3} \\ L_{b_4} \\ L_{b_5} \\ \hline L_{b_6} \end{array}\right]$, a super column vector of natural order 6×1. Clearly A is partitioned between the columns 2 and 3 and B is partitioned between the rows 2 and 3 and A is partitioned between the columns 5 and 6 and B is partitioned between the rows 5 and 6.

Now we find the product

$$AB = \left[L_{a_1} \quad L_{a_2} \mid L_{a_3} \quad L_{a_4} \quad L_{a_5} \mid L_{a_6}\right] \left[\begin{array}{c} L_{b_1} \\ L_{b_2} \\ \hline L_{b_3} \\ L_{b_4} \\ L_{b_5} \\ \hline L_{b_6} \end{array}\right]$$

$$= \begin{bmatrix} L_{a_1} & L_{a_2} \end{bmatrix} \begin{bmatrix} L_{b_1} \\ L_{b_2} \end{bmatrix} + \begin{bmatrix} L_{a_3} & L_{a_4} & L_{a_5} \end{bmatrix} \begin{bmatrix} L_{b_3} \\ L_{b_4} \\ L_{b_5} \end{bmatrix} + \begin{bmatrix} L_{a_6} \times L_{b_6} \end{bmatrix}$$

$$= \begin{bmatrix} L_{a_1}.L_{b_1} + L_{a_2}.L_{b_2} \end{bmatrix} + \begin{bmatrix} L_{a_3}.L_{b_3} + L_{a_4}.L_{b_4} + L_{a_5}.L_{b_5} \end{bmatrix} L_{a_6}.L_{b_6}$$

$$= \begin{bmatrix} L_{a_1b_1/(m+1)} + L_{a_2b_2/(m+1)} \end{bmatrix} \begin{bmatrix} L_{a_3b_3/(m+1)} + L_{a_4b_4/(m+1)} + L_{a_5b_5/(m+1)} \end{bmatrix}$$
$$+ L_{a_6b_6/(m+1)}$$

$$= \begin{bmatrix} L_{a_1b_1+a_2b_2/(m+1)} \end{bmatrix} + \begin{bmatrix} L_{(a_3b_3+a_4b_4+a_5b_5)/(m+1)} \end{bmatrix} + L_{a_6b_6/(m+1)}$$

$$= \begin{bmatrix} L_{(a_1b_1+a_2b_2+a_3b_3+a_4b_4+a_5b_5+a_6b_6)/(m+1)} \end{bmatrix} \text{ is in } L_R.$$

Suppose

$$A = \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ \hline L_{a_6} \\ L_{a_7} \\ L_{a_8} \\ L_{a_9} \end{array}\right]$$

and

$$B = \left[\begin{array}{c|c|c|ccc|ccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} \end{array}\right]$$

be super column vector and super row vector respectively of refined labels. BA is not defined though both A and B are of natural order 9 × 1 and 1 × 9 respectively as they have not the same type of partitions for we see A is partitioned between row three and four but B is partitioned between one and two columns and so on.

Let us consider

$$B = \left[L_{a_1} \quad L_{a_2} \mid L_{a_3} \quad L_{a_4} \right] \text{ and } A = \left[\begin{array}{c} L_{b_1} \\ L_{b_2} \\ \hline L_{b_3} \\ L_{b_4} \end{array} \right]$$

be super row vector and super column vector respectively of refined labels. We see

$$BA = \left[L_{a_1} \quad L_{a_2} \mid L_{a_3} \quad L_{a_4} \right] \left[\begin{array}{c} L_{b_1} \\ L_{b_2} \\ \hline L_{b_3} \\ L_{b_4} \end{array} \right]$$

$$= \left(L_{a_1}.L_{b_1} + L_{a_2} + L_{b_2} \right) \; \left(L_{a_3}.L_{b_3} + L_{a_4} + L_{b_4} \right)$$

$$= \left(L_{a_1 b_1/(m+1)} + L_{a_2 + b_2/(m+1)} \right) \mid \left(L_{a_3 b_3/(m+1)} + L_{a_4 + b_4/(m+1)} \right)$$

$$= L_{(a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4)/m+1}.$$

Let us now define product of a super column vector with a super row vector of refined labels.

Let

$$A = \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ L_{a_8} \end{array} \right] \text{ and } B = \left[L_{c_1} \quad L_{c_2} \mid L_{c_3} \quad L_{c_4} \quad L_{c_5} \mid L_{c_6} \right]$$

be a super column vector and super row vector respectively of refined labels. We see the product AB is defined even though they are of very different natural order and distinct in partition.

Consider

$$AB = \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ L_{a_8} \end{array}\right] \left[\begin{array}{cc|ccc|c} L_{c_1} & L_{c_2} & L_{c_3} & L_{c_4} & L_{c_5} & L_{c_6} \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} \begin{bmatrix} L_{a_1} \\ L_{a_2} \\ L_{a_3} \end{bmatrix} \begin{pmatrix} L_{c_1} & L_{c_2} \end{pmatrix} & \begin{bmatrix} L_{a_1} \\ L_{a_2} \\ L_{a_3} \end{bmatrix} \begin{pmatrix} L_{c_3} & L_{c_4} & L_{c_5} \end{pmatrix} & \begin{bmatrix} L_{a_1} \\ L_{a_2} \\ L_{a_3} \end{bmatrix} L_{c_6} \\ \hline L_{a_4} \begin{pmatrix} L_{c_1} & L_{c_2} \end{pmatrix} & L_{a_4} \begin{pmatrix} L_{c_3} & L_{c_4} & L_{c_5} \end{pmatrix} & L_{a_4}.L_{c_6} \\ \hline \begin{bmatrix} L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ L_{a_8} \end{bmatrix} \begin{pmatrix} L_{c_1} & L_{c_2} \end{pmatrix} & \begin{bmatrix} L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ L_{a_8} \end{bmatrix} \begin{pmatrix} L_{c_3} & L_{c_4} & L_{c_5} \end{pmatrix} & \begin{bmatrix} L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ L_{a_8} \end{bmatrix} L_{c_6} \end{array}\right]$$

$$
= \left[\begin{array}{cc|ccc|c}
L_{a_1}L_{c_1} & L_{a_1}L_{c_2} & L_{a_1}L_{c_3} & L_{a_1}L_{c_4} & L_{a_1}L_{c_5} & L_{a_1}L_{c_6} \\
L_{a_2}L_{c_1} & L_{a_2}L_{c_2} & L_{a_2}L_{c_3} & L_{a_2}L_{c_4} & L_{a_2}L_{c_5} & L_{a_2}L_{c_6} \\
L_{a_3}L_{c_1} & L_{a_3}L_{c_2} & L_{a_3}L_{c_3} & L_{a_3}L_{c_4} & L_{a_3}L_{c_5} & L_{a_3}L_{c_6} \\
\hline
L_{a_4}L_{c_1} & L_{a_4}L_{c_2} & L_{a_4}L_{c_3} & L_{a_4}L_{c_4} & L_{a_4}L_{c_5} & L_{a_4}L_{c_6} \\
\hline
L_{a_5}L_{c_1} & L_{a_5}L_{c_2} & L_{a_5}L_{c_3} & L_{a_5}L_{c_4} & L_{a_5}L_{c_5} & L_{a_5}L_{c_6} \\
L_{a_6}L_{c_1} & L_{a_6}L_{c_2} & L_{a_6}L_{c_3} & L_{a_6}L_{c_4} & L_{a_6}L_{c_5} & L_{a_6}L_{c_6} \\
L_{a_7}L_{c_1} & L_{a_7}L_{c_2} & L_{a_7}L_{c_3} & L_{a_7}L_{c_4} & L_{a_7}L_{c_5} & L_{a_7}L_{c_6} \\
L_{a_8}L_{c_1} & L_{a_8}L_{c_2} & L_{a_8}L_{c_3} & L_{a_8}L_{c_4} & L_{a_8}L_{c_5} & L_{a_8}L_{c_6}
\end{array}\right]
$$

$$
= \left[\begin{array}{cc|ccc|c}
L_{a_1c_1/m+1} & L_{a_1c_2/m+1} & L_{a_1c_3/m+1} & L_{a_1c_4/m+1} & L_{a_1c_5/m+1} & L_{a_1c_6/m+1} \\
L_{a_2c_1/m+1} & L_{a_2c_2/m+1} & L_{a_2c_3/m+1} & L_{a_2c_4/m+1} & L_{a_2c_5/m+1} & L_{a_2c_6/m+1} \\
L_{a_3c_1/m+1} & L_{a_3c_2/m+1} & L_{a_3c_3/m+1} & L_{a_3c_4/m+1} & L_{a_3c_5/m+1} & L_{a_3c_6/m+1} \\
\hline
L_{a_4c_1/m+1} & L_{a_4c_2/m+1} & L_{a_4c_3/m+1} & L_{a_4c_4/m+1} & L_{a_4c_5/m+1} & L_{a_4c_6/m+1} \\
\hline
L_{a_5c_1/m+1} & L_{a_5c_2/m+1} & L_{a_5c_3/m+1} & L_{a_5c_4/m+1} & L_{a_5c_5/m+1} & L_{a_5c_6/m+1} \\
L_{a_6c_1/m+1} & L_{a_6c_2/m+1} & L_{a_6c_3/m+1} & L_{a_6c_4/m+1} & L_{a_6c_5/m+1} & L_{a_6c_6/m+1} \\
L_{a_7c_1/m+1} & L_{a_7c_2/m+1} & L_{a_7c_3/m+1} & L_{a_7c_4/m+1} & L_{a_7c_5/m+1} & L_{a_7c_6/m+1} \\
L_{a_8c_1/m+1} & L_{a_8c_2/m+1} & L_{a_8c_3/m+1} & L_{a_8c_4/m+1} & L_{a_8c_5/m+1} & L_{a_8c_6/m+1}
\end{array}\right]
$$

is a super matrix of refined labels. We can multiply any super column vector with any other super row vector and the product is defined and the resultant is a super matrix of refined labels.

Let

$$C = \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \end{array}\right]$$

be a super column vector of refined labels.

$C^t = \left(L_{a_1} \mid L_{a_2} \quad L_{a_3} \mid L_{a_4} \quad L_{a_5} \quad L_{a_6} \mid L_{a_7} \quad L_{a_8}\right)$ be the transpose of C. Clearly C^t is a super row vector of refined labels.

We now find

$$CC^t = \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \end{array}\right] \left(L_{a_1} \mid L_{a_2} \quad L_{a_3} \mid L_{a_4} \quad L_{a_5} \quad L_{a_6} \mid L_{a_7} \quad L_{a_8}\right)$$

$$
=\left[\begin{array}{c|c|c|c}
L_{a_1}L_{a_1} & L_{a_1}\begin{pmatrix}L_{a_2} & L_{a_3}\end{pmatrix} & L_{a_1}\begin{pmatrix}L_{a_4} & L_{a_5} & L_{a_6}\end{pmatrix} & L_{a_1}\begin{pmatrix}L_{a_7} & L_{a_8}\end{pmatrix} \\
\hline
\begin{bmatrix}L_{a_2}\\ L_{a_3}\end{bmatrix}\begin{pmatrix}L_{a_1}\end{pmatrix} & \begin{bmatrix}L_{a_2}\\ L_{a_3}\end{bmatrix}\begin{pmatrix}L_{a_2} & L_{a_3}\end{pmatrix} & \begin{bmatrix}L_{a_2}\\ L_{a_3}\end{bmatrix}\begin{pmatrix}L_{a_4} & L_{a_5} & L_{a_6}\end{pmatrix} & \begin{bmatrix}L_{a_2}\\ L_{a_3}\end{bmatrix}\begin{pmatrix}L_{a_7} & L_{a_8}\end{pmatrix} \\
\hline
\begin{bmatrix}L_{a_4}\\ L_{a_5}\\ L_{a_6}\end{bmatrix}\begin{pmatrix}L_{a_1}\end{pmatrix} & \begin{bmatrix}L_{a_4}\\ L_{a_5}\\ L_{a_6}\end{bmatrix}\begin{pmatrix}L_{a_2} & L_{a_3}\end{pmatrix} & \begin{bmatrix}L_{a_4}\\ L_{a_5}\\ L_{a_6}\end{bmatrix}\begin{pmatrix}L_{a_4} & L_{a_5} & L_{a_6}\end{pmatrix} & \begin{bmatrix}L_{a_4}\\ L_{a_5}\\ L_{a_6}\end{bmatrix}\begin{pmatrix}L_{a_7} & L_{a_8}\end{pmatrix} \\
\hline
\begin{bmatrix}L_{a_7}\\ L_{a_8}\end{bmatrix}\begin{pmatrix}L_{a_1}\end{pmatrix} & \begin{bmatrix}L_{a_7}\\ L_{a_8}\end{bmatrix}\begin{pmatrix}L_{a_2} & L_{a_3}\end{pmatrix} & \begin{bmatrix}L_{a_7}\\ L_{a_8}\end{bmatrix}\begin{pmatrix}L_{a_4} & L_{a_5} & L_{a_6}\end{pmatrix} & \begin{bmatrix}L_{a_7}\\ L_{a_8}\end{bmatrix}\begin{pmatrix}L_{a_7} & L_{a_8}\end{pmatrix}
\end{array}\right]
$$

$$
=\left[\begin{array}{c|cc|ccc|cc}
L_{a_1}L_{a_1} & L_{a_1}L_{a_2} & L_{a_1}L_{a_3} & L_{a_1}L_{a_4} & L_{a_1}L_{a_5} & L_{a_1}L_{a_6} & L_{a_1}L_{a_7} & L_{a_1}L_{a_8} \\
\hline
L_{a_2}L_{a_1} & L_{a_2}L_{a_2} & L_{a_2}L_{a_3} & L_{a_2}L_{a_4} & L_{a_2}L_{a_5} & L_{a_2}L_{a_6} & L_{a_2}L_{a_7} & L_{a_2}L_{a_8} \\
L_{a_3}L_{a_1} & L_{a_3}L_{a_2} & L_{a_3}L_{a_3} & L_{a_3}L_{a_4} & L_{a_3}L_{a_5} & L_{a_3}L_{a_6} & L_{a_3}L_{a_7} & L_{a_3}L_{a_8} \\
\hline
L_{a_4}L_{a_1} & L_{a_5}L_{a_2} & L_{a_4}L_{a_3} & L_{a_4}L_{a_4} & L_{a_4}L_{a_5} & L_{a_4}L_{a_6} & L_{a_4}L_{a_7} & L_{a_4}L_{a_8} \\
L_{a_5}L_{a_1} & L_{a_5}L_{a_2} & L_{a_5}L_{a_3} & L_{a_5}L_{a_4} & L_{a_5}L_{a_5} & L_{a_5}L_{a_6} & L_{a_5}L_{a_7} & L_{a_5}L_{a_8} \\
L_{a_6}L_{a_1} & L_{a_6}L_{a_2} & L_{a_6}L_{a_3} & L_{a_6}L_{a_4} & L_{a_6}L_{a_5} & L_{a_6}L_{a_6} & L_{a_6}L_{a_7} & L_{a_6}L_{a_8} \\
\hline
L_{a_7}L_{a_1} & L_{a_7}L_{a_2} & L_{a_7}L_{a_3} & L_{a_7}L_{a_4} & L_{a_7}L_{a_5} & L_{a_7}L_{a_6} & L_{a_7}L_{a_7} & L_{a_7}L_{a_8} \\
L_{a_8}L_{a_1} & L_{a_8}L_{a_2} & L_{a_8}L_{a_3} & L_{a_8}L_{a_4} & L_{a_8}L_{a_5} & L_{a_8}L_{a_6} & L_{a_8}L_{a_7} & L_{a_8}L_{a_8}
\end{array}\right]
$$

$$
= \left[\begin{array}{c|cc|ccc|cc}
L_{a_1^2/m+1} & L_{a_1a_2/m+1} & L_{a_1a_3/m+1} & L_{a_1a_4/m+1} & L_{a_1a_5/m+1} & L_{a_1a_6/m+1} & L_{a_1a_7/m+1} & L_{a_1a_8/m+1} \\
\hline
L_{a_2a_1/m+1} & L_{a_2a_2/m+1} & L_{a_2a_3/m+1} & L_{a_2a_4/m+1} & L_{a_2a_5/m+1} & L_{a_2a_6/m+1} & L_{a_2a_7/m+1} & L_{a_2a_8/m+1} \\
L_{a_3a_1/m+1} & L_{a_3a_2/m+1} & L_{a_3a_3/m+1} & L_{a_3a_4/m+1} & L_{a_3a_5/m+1} & L_{a_3a_6/m+1} & L_{a_3a_7/m+1} & L_{a_3a_8/m+1} \\
\hline
L_{a_4a_1/m+1} & L_{a_5a_2/m+1} & L_{a_4a_3/m+1} & L_{a_4a_4/m+1} & L_{a_4a_5/m+1} & L_{a_4a_6/m+1} & L_{a_4a_7/m+1} & L_{a_4a_8/m+1} \\
L_{a_5a_1/m+1} & L_{a_5a_2/m+1} & L_{a_5a_3/m+1} & L_{a_5a_4/m+1} & L_{a_5a_5/m+1} & L_{a_5a_6/m+1} & L_{a_5a_7/m+1} & L_{a_5a_8/m+1} \\
L_{a_6a_1/m+1} & L_{a_6a_2/m+1} & L_{a_6a_3/m+1} & L_{a_6a_4/m+1} & L_{a_6a_5/m+1} & L_{a_6a_6/m+1} & L_{a_6a_7/m+1} & L_{a_6a_8/m+1} \\
\hline
L_{a_7a_1/m+1} & L_{a_7a_2/m+1} & L_{a_7a_3/m+1} & L_{a_7a_4/m+1} & L_{a_7a_5/m+1} & L_{a_7a_6/m+1} & L_{a_7a_7/m+1} & L_{a_7a_8/m+1} \\
L_{a_8a_1/m+1} & L_{a_8a_2/m+1} & L_{a_8a_3/m+1} & L_{a_8a_4/m+1} & L_{a_8a_5/m+1} & L_{a_8a_6/m+1} & L_{a_8a_7/m+1} & L_{a_8a_8/m+1}
\end{array}\right]
$$

is a super matrix of refined labels, we see the super matrix is a symmetric matrix of refined labels. Thus we get by multiplying the column super vector with its transpose the symmetric matrix of refined labels.

We will illustrate by some more examples.

Let

$$
P = \left[\begin{array}{c}
L_{a_1} \\
L_{a_2} \\
\hline
L_{a_3} \\
L_{a_4} \\
L_{a_5} \\
\hline
L_{a_6}
\end{array}\right]
$$

be a super column vector of refined labels. $P^t = \left[\begin{array}{cc|ccc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} \end{array}\right]$ be the transpose of P.

$$
P.P^t = \left[\begin{array}{c}
L_{a_1} \\
L_{a_2} \\
\hline
L_{a_3} \\
L_{a_4} \\
L_{a_5} \\
\hline
L_{a_6}
\end{array}\right]
\left[\begin{array}{cc|ccc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} \end{array}\right]
$$

$$= \left[\begin{array}{c|c|c}
\begin{bmatrix} L_{a_1} \\ L_{a_2} \end{bmatrix}\begin{bmatrix} L_{a_1} & L_{a_2} \end{bmatrix} & \begin{bmatrix} L_{a_1} \\ L_{a_2} \end{bmatrix}\begin{bmatrix} L_{a_3} & L_{a_4} & L_{a_5} \end{bmatrix} & \begin{bmatrix} L_{a_1} \\ L_{a_2} \end{bmatrix} L_{a_6} \\
\hline
\begin{bmatrix} L_{a_3} \\ L_{a_4} \\ L_{a_5} \end{bmatrix}\begin{bmatrix} L_{a_1} & L_{a_2} \end{bmatrix} & \begin{bmatrix} L_{a_3} \\ L_{a_4} \\ L_{a_5} \end{bmatrix}\begin{bmatrix} L_{a_3} & L_{a_4} & L_{a_5} \end{bmatrix} & \begin{bmatrix} L_{a_3} \\ L_{a_4} \\ L_{a_5} \end{bmatrix} L_{a_6} \\
\hline
L_{a_6}\begin{bmatrix} L_{a_1} & L_{a_2} \end{bmatrix} & L_{a_6}\begin{bmatrix} L_{a_3} & L_{a_4} & L_{a_5} \end{bmatrix} & L_{a_6}.L_{a_6}
\end{array}\right]$$

$$= \left[\begin{array}{cc|ccc|c}
L_{a_1}L_{a_1} & L_{a_1}L_{a_2} & L_{a_1}L_{a_3} & L_{a_1}L_{a_4} & L_{a_1}L_{a_5} & L_{a_1}L_{a_6} \\
L_{a_2}L_{a_1} & L_{a_2}L_{a_2} & L_{a_2}L_{a_3} & L_{a_2}L_{a_4} & L_{a_2}L_{a_5} & L_{a_2}L_{a_6} \\
\hline
L_{a_3}L_{a_1} & L_{a_3}L_{a_2} & L_{a_3}L_{a_3} & L_{a_3}L_{a_4} & L_{a_3}L_{a_5} & L_{a_3}L_{a_6} \\
L_{a_4}L_{a_1} & L_{a_4}L_{a_2} & L_{a_4}L_{a_3} & L_{a_4}L_{a_4} & L_{a_4}L_{a_5} & L_{a_4}L_{a_6} \\
L_{a_5}L_{a_1} & L_{a_5}L_{a_2} & L_{a_5}L_{a_3} & L_{a_5}L_{a_4} & L_{a_5}L_{a_5} & L_{a_5}L_{a_6} \\
\hline
L_{a_6}L_{a_1} & L_{a_6}L_{a_2} & L_{a_6}L_{a_3} & L_{a_6}L_{a_4} & L_{a_6}L_{a_5} & L_{a_6}L_{a_6}
\end{array}\right]$$

$$= \left[\begin{array}{cc|ccc|c}
L_{a_1^2/m+1} & L_{a_1a_2/m+1} & L_{a_1a_3/m+1} & L_{a_1a_4/m+1} & L_{a_1a_5/m+1} & L_{a_1a_6/m+1} \\
L_{a_2a_1/m+1} & L_{a_2^2/m+1} & L_{a_2a_3/m+1} & L_{a_2a_4/m+1} & L_{a_2a_5/m+1} & L_{a_2a_6/m+1} \\
\hline
L_{a_3a_1/m+1} & L_{a_3a_2/m+1} & L_{a_3^2/m+1} & L_{a_3a_4/m+1} & L_{a_3a_5/m+1} & L_{a_3a_6/m+1} \\
L_{a_4a_1/m+1} & L_{a_4a_2/m+1} & L_{a_4a_3/m+1} & L_{a_4^2/m+1} & L_{a_4a_5/m+1} & L_{a_4a_6/m+1} \\
L_{a_5a_1/m+1} & L_{a_5a_2/m+1} & L_{a_5a_3/m+1} & L_{a_5a_4/m+1} & L_{a_5^2/m+1} & L_{a_5a_6/m+1} \\
\hline
L_{a_6a_1/m+1} & L_{a_6a_2/m+1} & L_{a_6a_3/m+1} & L_{a_6a_4/m+1} & L_{a_6a_5/m+1} & L_{a_6^2/m+1}
\end{array}\right]$$

is a symmetric super matrix of refined labels.

Let

$$A = \left[\begin{array}{cc|c|ccc} L_{a_1} & L_{a_2} & L_{a_7} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_3} & L_{a_4} & L_{a_8} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_5} & L_{a_6} & L_{a_9} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \end{array}\right]$$

be a super matrix of row vectors of refined labels.

$$A^t = \left[\begin{array}{ccc} L_{a_1} & L_{a_3} & L_{a_5} \\ L_{a_2} & L_{a_4} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{13}} & L_{a_{16}} \\ L_{a_{11}} & L_{a_{14}} & L_{a_{17}} \\ L_{a_{12}} & L_{a_{15}} & L_{a_{18}} \end{array}\right]$$

be the transpose the super row vectors. A of refined labels. To find

$$A.A^t = \left[\begin{array}{cc|c|ccc} L_{a_1} & L_{a_2} & L_{a_7} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_3} & L_{a_4} & L_{a_8} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_5} & L_{a_6} & L_{a_9} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \end{array}\right] \left[\begin{array}{ccc} L_{a_1} & L_{a_3} & L_{a_5} \\ L_{a_2} & L_{a_4} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{13}} & L_{a_{16}} \\ L_{a_{11}} & L_{a_{14}} & L_{a_{17}} \\ L_{a_{12}} & L_{a_{15}} & L_{a_{18}} \end{array}\right]$$

$$= \begin{bmatrix} L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \end{bmatrix} \begin{bmatrix} L_{a_1} & L_{a_3} & L_{a_5} \\ L_{a_2} & L_{a_4} & L_{a_6} \end{bmatrix} + \begin{bmatrix} L_{a_7} \\ L_{a_8} \\ L_{a_9} \end{bmatrix} \begin{bmatrix} L_{a_7} & L_{a_8} & L_{a_9} \end{bmatrix}$$

$$+ \begin{bmatrix} L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \end{bmatrix} \begin{bmatrix} L_{a_{10}} & L_{a_{13}} & L_{a_{16}} \\ L_{a_{11}} & L_{a_{14}} & L_{a_{17}} \\ L_{a_{12}} & L_{a_{15}} & L_{a_{18}} \end{bmatrix}$$

$$
= \begin{bmatrix} L_{a_1}L_{a_1} + L_{a_2}L_{a_2} & L_{a_1}L_{a_3} + L_{a_2}L_{a_4} & L_{a_1}L_{a_5} + L_{a_2}L_{a_6} \\ L_{a_3}L_{a_1} + L_{a_4}L_{a_2} & L_{a_3}L_{a_3} + L_{a_4}L_{a_4} & L_{a_3}L_{a_5} + L_{a_4}L_{a_6} \\ L_{a_5}L_{a_1} + L_{a_6}L_{a_2} & L_{a_5}L_{a_3} + L_{a_6}L_{a_4} & L_{a_5}L_{a_5} + L_{a_6}L_{a_6} \end{bmatrix}
$$

$$
+ \begin{bmatrix} L_{a_7}L_{a_7} & L_{a_7}L_{a_8} & L_{a_7}L_{a_9} \\ L_{a_8}L_{a_7} & L_{a_8}L_{a_8} & L_{a_8}L_{a_9} \\ L_{a_9}L_{a_7} & L_{a_9}L_{a_8} & L_{a_9}L_{a_9} \end{bmatrix}
$$

$$
+ \begin{bmatrix} L_{a_{10}}L_{a_{10}} + L_{a_{11}}L_{a_{11}} + L_{a_{12}}L_{a_{12}} & L_{a_{10}}L_{a_{13}} + L_{a_{11}}L_{a_{14}} + L_{a_{12}}L_{a_{15}} & L_{a_{10}}L_{a_{16}} + L_{a_{11}}L_{a_{17}} + L_{a_{12}}L_{a_{18}} \\ L_{a_{13}}L_{a_{10}} + L_{a_{14}}L_{a_{11}} + L_{a_{15}}L_{a_{12}} & L_{a_{13}}L_{a_{13}} + L_{a_{14}}L_{a_{14}} + L_{a_{15}}L_{a_{15}} & L_{a_{13}}L_{a_{16}} + L_{a_{14}}L_{a_{17}} + L_{a_{15}}L_{a_{18}} \\ L_{a_{16}}L_{a_{10}} + L_{a_{17}}L_{a_{11}} + L_{a_{18}}L_{a_{12}} & L_{a_{16}}L_{a_{13}} + L_{a_{17}}L_{a_{14}} + L_{a_{18}}L_{a_{15}} & L_{a_{16}}L_{a_{16}} + L_{a_{17}}L_{a_{17}} + L_{a_{18}}L_{a_{18}} \end{bmatrix}
$$

$$
= \begin{bmatrix} L_{a_1^2/m+1} + L_{a_2^2/m+1} & L_{a_1 a_3/m+1} + L_{a_2 a_4/m+1} & L_{a_1 a_5/m+1} + L_{a_2 a_6/m+1} \\ L_{a_3 a_1/m+1} + L_{a_4 a_2/m+1} & L_{a_3^2/m+1} + L_{a_4^2/m+1} & L_{a_3 a_5/m+1} + L_{a_4 a_6/m+1} \\ L_{a_5 a_1/m+1} + L_{a_6 a_2/m+1} & L_{a_5 a_3/m+1} + L_{a_6 a_4/m+1} & L_{a_5^2/m+1} + L_{a_6^2/m+1} \end{bmatrix}
$$

$$
+ \begin{bmatrix} L_{a_7^2/m+1} & L_{a_7 a_8/m+1} & L_{a_7 a_9/m+1} \\ L_{a_8 a_7/m+1} & L_{a_8^2/m+1} & L_{a_8 a_9/m+1} \\ L_{a_9 a_7/m+1} & L_{a_9 a_8/m+1} & L_{a_9^2/m+1} \end{bmatrix} +
$$

$$
\begin{bmatrix} L_{a_{10}^2/m+1} + L_{a_{11}^2/m+1} + L_{a_{12}^2/m+1} & L_{a_{10}a_{13}/m+1} + L_{a_{11}a_{14}/m+1} + L_{a_{12}a_{15}/m+1} & L_{a_{10}a_{16}/m+1} + L_{a_{11}a_{17}/m+1} + L_{a_{12}a_{18}/m+1} \\ L_{a_{13}a_{10}/m+1} + L_{a_{14}a_{11}/m+1} + L_{a_{15}a_{12}/m+1} & L_{a_{13}^2/m+1} + L_{a_{14}^2/m+1} + L_{a_{15}^2/m+1} & L_{a_{13}a_{16}/m+1} + L_{a_{14}a_{17}/m+1} + L_{a_{15}a_{18}/m+1} \\ L_{a_{16}a_{10}/m+1} + L_{a_{17}a_{11}/m+1} + L_{a_{18}a_{12}/m+1} & L_{a_{16}a_{13}/m+1} + L_{a_{17}a_{14}/m+1} + L_{a_{18}a_{15}/m+1} & L_{a_{16}^2/m+1} + L_{a_{17}^2/m+1} + L_{a_{18}^2/m+1} \end{bmatrix}
$$

$$
= \left[\begin{matrix} L_{a_1^2 + a_2^2 + a_7^2 + a_{10}^2 + a_{11}^2 + a_{12}^2 /m+1} & L_{a_1 a_3 + a_2 a_4 + a_7 a_8 + a_{10}a_{13} + a_{11}a_{14} + a_{12}a_{15}/m+1} \\ L_{a_3 a_1 + a_4 a_2 + a_8 a_7 + a_{13}a_{10} + a_{14}a_{11} + a_{15}a_{12}/m+1} & L_{a_3^2 + a_4^2 + a_8^2 + a_{13}^2 + a_{14}^2 + a_{15}^2 /m+1} \\ L_{a_5 a_1 + a_6 a_2 + a_9 a_7 + a_{16}a_{10} + a_{17}a_{11} + a_{18}a_{12}/m+1} & L_{a_5 a_3 + a_6 a_4 + a_9 a_8 + a_{16}a_{13} + a_{17}a_{14} + a_{18}a_{15}/m+1} \end{matrix} \right.
$$

$$\left.\begin{matrix} L_{a_1 a_5 + a_2 a_6 + a_7 a_9 + a_{10} a_{16} + a_{11} a_{17} + a_{12} a_{18} / m+1} \\ L_{a_3 a_5 + a_4 a_6 + a_8 a_9 + a_{13} a_{16} + a_{14} a_{17} + a_{15} a_{18} / m+1} \\ L_{a_5^2 + a_6^2 + a_9^2 + a_{16}^2 + a_{17}^2 + a_{18}^2 / m+1} \end{matrix}\right].$$

We see AA^t is a ordinary matrix of refined labels. Clearly AA^t is a symmetric matrix of refined labels. Thus using a product of super row vectors with its transpose we can get a symmetric matrix of refined labels.

Consider

$$H = \left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} \\ \hline L_{a_{11}} & L_{a_{12}} \end{array}\right]$$

a super column vector of refined labels.

Clearly

$$H^t = \left[\begin{array}{ccc|cc|c} L_{a_1} & L_{a_3} & L_{a_5} & L_{a_7} & L_{a_9} & L_{a_{11}} \\ L_{a_2} & L_{a_4} & L_{a_6} & L_{a_8} & L_{a_{10}} & L_{a_{12}} \end{array}\right]$$

is the transpose of H which is a super row vector of refined labels.

Now we find

$$H^T H = \left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} \\ \hline L_{a_{11}} & L_{a_{12}} \end{array}\right] \times \left[\begin{array}{ccc|cc|c} L_{a_1} & L_{a_3} & L_{a_5} & L_{a_7} & L_{a_9} & L_{a_{11}} \\ L_{a_2} & L_{a_4} & L_{a_6} & L_{a_8} & L_{a_{10}} & L_{a_{12}} \end{array}\right]$$

$$= \begin{bmatrix} L_{a_1} & L_{a_3} & L_{a_5} \\ L_{a_2} & L_{a_4} & L_{a_6} \end{bmatrix} \begin{bmatrix} L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \end{bmatrix} +$$

$$\begin{bmatrix} L_{a_7} & L_{a_9} \\ L_{a_8} & L_{a_{10}} \end{bmatrix} \begin{bmatrix} L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} \end{bmatrix} + \begin{bmatrix} L_{a_{11}} \\ L_{a_{12}} \end{bmatrix} \begin{bmatrix} L_{a_{11}} & L_{a_{12}} \end{bmatrix}$$

$$= \begin{bmatrix} L_{a_1}L_{a_1} + L_{a_3}L_{a_3} + L_{a_5}L_{a_5} & L_{a_1}L_{a_2} + L_{a_3}L_{a_4} + L_{a_5}L_{a_6} \\ L_{a_2}L_{a_1} + L_{a_4}L_{a_3} + L_{a_6}L_{a_5} & L_{a_2}L_{a_2} + L_{a_4}L_{a_4} + L_{a_6}L_{a_6} \end{bmatrix} +$$

$$\begin{bmatrix} L_{a_7}L_{a_7} + L_{a_9}L_{a_9} & L_{a_7}L_{a_8} + L_{a_9}L_{a_{10}} \\ L_{a_8}L_{a_7} + L_{a_{10}}L_{a_9} & L_{a_8}L_{a_8} + L_{a_{10}}L_{a_{10}} \end{bmatrix}$$

$$+ \begin{bmatrix} L_{a_{11}}L_{a_{11}} & L_{a_{11}}L_{a_{12}} \\ L_{a_{12}}L_{a_{11}} & L_{a_{12}}L_{a_{12}} \end{bmatrix}$$

$$= \begin{bmatrix} L_{a_1^2/m+1} + L_{a_3^2/m+1} + L_{a_5^2/m+1} & L_{a_1a_2/m+1} + L_{a_3a_4/m+1} + L_{a_5a_6/m+1} \\ L_{a_2a_1/m+1} + L_{a_4a_3/m+1} + L_{a_6a_5/m+1} & L_{a_2^2/m+1} + L_{a_4^2/m+1} + L_{a_6^2/m+1} \end{bmatrix}$$

$$+ \begin{bmatrix} L_{a_7^2/m+1} + L_{a_9^2/m+1} & L_{a_7a_8/m+1} + L_{a_9a_{10}/m+1} \\ L_{a_8a_7/m+1} + L_{a_{10}a_9/m+1} & L_{a_8^2/m+1} + L_{a_{10}^2/m+1} \end{bmatrix}$$

$$+ \begin{bmatrix} L_{a_{11}^2/m+1} & L_{a_{11}a_{12}/m+1} \\ L_{a_{12}a_{11}/m+1} & L_{a_{12}^2/m+1} \end{bmatrix}$$

$$= \begin{bmatrix} L_{a_1^2+a_3^2+a_5^2+a_7^2+a_9^2+a_{11}^2/m+1} & L_{a_1a_2+a_3a_4+a_5a_6+a_7a_8+a_9a_{10}+a_{11}a_{12}/m+1} \\ L_{a_2a_1+a_4a_3+a_6a_5+a_8a_7+a_{10}a_9+a_{12}a_{11}/m+1} & L_{a_2^2+a_4^2+a_6^2+a_8^2+a_{10}^2+a_{12}^2/m+1} \end{bmatrix}$$

is a symmetric matrix of refined labels which is not a super matrix of refined labels.

Thus in applications we can multiply two super matrices of refined labels and get a symmetric matrix of refined labels,

likewise we can multiply super matrices to get back super matrices.

For consider

$$X = \left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} \end{array}\right] \text{ and } Y = \left[\begin{array}{c|cc} L_{b_1} & L_{b_3} & L_{b_4} \\ L_{b_2} & L_{b_5} & L_{b_6} \end{array}\right]$$

two super matrices of refined labels. Now we find

$$XY = \left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} \end{array}\right] \left[\begin{array}{c|cc} L_{b_1} & L_{b_3} & L_{b_4} \\ L_{b_2} & L_{b_5} & L_{b_6} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \begin{bmatrix} L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \end{bmatrix} \begin{bmatrix} L_{b_1} \\ L_{b_2} \end{bmatrix} & \begin{bmatrix} L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \end{bmatrix} \begin{bmatrix} L_{b_3} & L_{b_4} \\ L_{b_5} & L_{b_6} \end{bmatrix} \\ \hline \begin{pmatrix} L_{a_5} & L_{a_6} \end{pmatrix} \begin{bmatrix} L_{b_1} \\ L_{b_2} \end{bmatrix} & \begin{pmatrix} L_{a_5} & L_{a_6} \end{pmatrix} \begin{bmatrix} L_{b_3} & L_{b_4} \\ L_{b_5} & L_{b_6} \end{bmatrix} \\ \hline \begin{bmatrix} L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} \end{bmatrix} \begin{bmatrix} L_{b_1} \\ L_{b_2} \end{bmatrix} & \begin{bmatrix} L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} \end{bmatrix} \begin{bmatrix} L_{b_3} & L_{b_4} \\ L_{b_5} & L_{b_6} \end{bmatrix} \end{array}\right]$$

$$= \left[\begin{array}{c|cc} L_{a_1}L_{b_1} + L_{a_2}L_{b_2} & L_{a_1}L_{b_3} + L_{a_2}L_{b_5} & L_{a_1}L_{b_4} + L_{a_2}L_{b_6} \\ L_{a_3}L_{b_1} + L_{a_4}L_{b_2} & L_{a_3}L_{b_3} + L_{a_4}L_{b_5} & L_{a_3}L_{b_4} + L_{a_4}L_{b_6} \\ \hline L_{a_5}L_{b_1} + L_{a_6}L_{b_2} & L_{a_5}L_{b_3} + L_{a_6}L_{b_5} & L_{a_5}L_{b_4} + L_{a_6}L_{b_6} \\ \hline L_{a_7}L_{b_1} + L_{a_8}L_{b_2} & L_{a_7}L_{b_3} + L_{a_8}L_{b_5} & L_{a_7}L_{b_4} + L_{a_8}L_{b_6} \\ L_{a_9}L_{b_1} + L_{a_{10}}L_{b_2} & L_{a_9}L_{b_3} + L_{a_{10}}L_{b_5} & L_{a_9}L_{b_4} + L_{a_{10}}L_{b_6} \\ L_{a_{11}}L_{b_1} + L_{a_{12}}L_{b_2} & L_{a_{11}}L_{b_3} + L_{a_{12}}L_{b_5} & L_{a_{11}}L_{b_4} + L_{a_{12}}L_{b_6} \end{array}\right]$$

$$= \left[\begin{array}{c|cc} L_{a_1 b_1 + a_2 b_2 / m+1} & L_{a_1 b_3 + a_2 b_5 / m+1} & L_{a_1 b_4 + a_2 b_6 / m+1} \\ L_{a_3 b_1 + a_4 b_2 / m+1} & L_{a_3 b_3 + a_4 b_5 / m+1} & L_{a_3 b_4 + a_4 b_6 / m+1} \\ \hline L_{a_5 b_1 + a_6 b_2 / m+1} & L_{a_5 b_3 + a_6 b_5 / m+1} & L_{a_5 b_4 + a_6 b_6 / m+1} \\ \hline L_{a_7 b_1 + a_8 b_2 / m+1} & L_{a_7 b_3 + a_8 b_5 / m+1} & L_{a_7 b_4 + a_8 b_6 / m+1} \\ L_{a_9 b_1 + a_{10} b_2 / m+1} & L_{a_9 b_3 + a_{10} b_5 / m+1} & L_{a_9 b_4 + a_{10} b_6 / m+1} \\ L_{a_{11} b_1 + a_{12} b_2 / m+1} & L_{a_{11} b_3 + a_{12} b_5 / m+1} & L_{a_{11} b_4 + a_{12} b_6 / m+1} \end{array}\right]$$

is a super matrix of refined labels.

We find

$$Y^t X^t = \left[\begin{array}{cc} L_{b_1} & L_{b_2} \\ \hline L_{b_3} & L_{b_4} \\ L_{b_5} & L_{b_6} \end{array}\right]$$

$$\left[\begin{array}{cc|c|ccc} L_{a_1} & L_{a_3} & L_{a_5} & L_{a_7} & L_{a_9} & L_{a_{11}} \\ L_{a_2} & L_{a_4} & L_{a_6} & L_{a_8} & L_{a_{10}} & L_{a_{12}} \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} \begin{pmatrix} L_{b_1} & L_{b_2} \end{pmatrix} \begin{bmatrix} L_{a_1} & L_{a_3} \\ L_{a_2} & L_{a_4} \end{bmatrix} & \begin{pmatrix} L_{b_1} & L_{b_2} \end{pmatrix} \begin{bmatrix} L_{a_5} \\ L_{a_6} \end{bmatrix} & \begin{pmatrix} L_{b_1} & L_{b_2} \end{pmatrix} \begin{bmatrix} L_{a_7} & L_{a_9} & L_{a_{11}} \\ L_{a_8} & L_{a_{10}} & L_{a_{12}} \end{bmatrix} \\ \hline \begin{bmatrix} L_{b_3} & L_{b_5} \\ L_{b_4} & L_{b_6} \end{bmatrix} \begin{bmatrix} L_{a_1} & L_{a_3} \\ L_{a_2} & L_{a_4} \end{bmatrix} & \begin{bmatrix} L_{b_3} & L_{b_5} \\ L_{b_4} & L_{b_6} \end{bmatrix} \begin{bmatrix} L_{a_5} \\ L_{a_6} \end{bmatrix} & \begin{bmatrix} L_{b_3} & L_{b_5} \\ L_{b_4} & L_{b_6} \end{bmatrix} \begin{bmatrix} L_{a_7} & L_{a_9} & L_{a_{11}} \\ L_{a_8} & L_{a_{10}} & L_{a_{12}} \end{bmatrix} \end{array}\right]$$

$$= \left[\begin{array}{cc|c}
L_{b_1}L_{a_1} + L_{b_2}L_{a_2} & L_{b_1}L_{a_3} + L_{b_2}L_{a_4} & L_{b_1}L_{a_5} + L_{b_2}L_{a_6} \\
\hline
L_{b_3}L_{a_1} + L_{b_5}L_{a_2} & L_{b_3}L_{a_3} + L_{b_5}L_{a_4} & L_{b_3}L_{a_5} + L_{b_5}L_{a_6} \\
L_{b_4}L_{a_1} + L_{b_6}L_{a_2} & L_{b_4}L_{a_3} + L_{b_6}L_{a_4} & L_{b_4}L_{a_5} + L_{b_6}L_{a_6} \\
L_{b_1}L_{a_7} + L_{b_2}L_{a_8} & L_{b_1}L_{a_9} + L_{b_2}L_{a_{10}} & L_{b_1}L_{a_{11}} + L_{b_2}L_{a_{12}} \\
\hline
L_{b_3}L_{a_7} + L_{b_5}L_{a_8} & L_{b_3}L_{a_9} + L_{b_5}L_{a_{10}} & L_{b_3}L_{a_{11}} + L_{b_5}L_{a_{12}} \\
L_{b_4}L_{a_7} + L_{b_6}L_{a_8} & L_{b_4}L_{a_9} + L_{b_6}L_{a_{10}} & L_{b_4}L_{a_{11}} + L_{b_6}L_{a_{12}}
\end{array}\right]$$

$$= \left[\begin{array}{cc|c}
L_{b_1 a_1 + b_2 a_2 / m+1} & L_{b_1 a_3 + b_2 a_4 / m+1} & L_{b_1 a_5 + b_2 a_6 / m+1} \\
\hline
L_{b_3 a_1 + b_5 a_2 / m+1} & L_{b_3 a_3 + b_5 a_4 / m+1} & L_{b_3 a_5 + b_5 a_6 / m+1} \\
L_{b_4 a_1 + b_6 a_2 / m+1} & L_{b_4 a_3 + b_6 a_4 / m+1} & L_{b_4 a_5 + b_6 a_6 / m+1} \\
L_{b_1 a_7 + b_2 a_8 / m+1} & L_{b_1 a_9 + b_2 a_{10} / m+1} & L_{b_1 a_{11} + b_2 a_{12} / m+1} \\
\hline
L_{b_3 a_7 + b_5 a_8 / m+1} & L_{b_3 a_9 + b_5 a_{10} / m+1} & L_{b_3 a_{11} + b_5 a_{12} / m+1} \\
L_{b_4 a_7 + b_6 a_8 / m+1} & L_{b_4 a_9 + b_6 a_{10} / m+1} & L_{b_4 a_{11} + b_6 a_{12} / m+1}
\end{array}\right].$$

We see $XY = Y^t X^t$ are super matrix of refined labels.

Now we proceed onto define another type of product of super matrix of refined labels.

Let

$$M = \left[\begin{array}{c|cc|ccc}
L_{a_1} & L_{a_8} & L_{a_{15}} & L_{a_{22}} & L_{a_{29}} & L_{a_{36}} \\
L_{a_2} & L_{a_9} & L_{a_{16}} & L_{a_{23}} & L_{a_{30}} & L_{a_{37}} \\
\hline
L_{a_3} & L_{a_{10}} & L_{a_{17}} & L_{a_{24}} & L_{a_{31}} & L_{a_{38}} \\
L_{a_4} & L_{a_{11}} & L_{a_{18}} & L_{a_{25}} & L_{a_{32}} & L_{a_{39}} \\
L_{a_5} & L_{a_{12}} & L_{a_{19}} & L_{a_{26}} & L_{a_{33}} & L_{a_{40}} \\
L_{a_6} & L_{a_{13}} & L_{a_{20}} & L_{a_{27}} & L_{a_{34}} & L_{a_{41}} \\
\hline
L_{a_7} & L_{a_{14}} & L_{a_{21}} & L_{a_{28}} & L_{a_{35}} & L_{a_{42}}
\end{array}\right]$$

be a super matrix of refined labels.

Consider

$$M^T = \left[\begin{array}{cc|cccc|c}
L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} \\ \hline
L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} \\
L_{a_{15}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \\ \hline
L_{a_{22}} & L_{a_{23}} & L_{a_{24}} & L_{a_{25}} & L_{a_{26}} & L_{a_{27}} & L_{a_{28}} \\
L_{a_{29}} & L_{a_{30}} & L_{a_{31}} & L_{a_{32}} & L_{a_{33}} & L_{a_{34}} & L_{a_{35}} \\
L_{a_{36}} & L_{a_{37}} & L_{a_{38}} & L_{a_{39}} & L_{a_{40}} & L_{a_{41}} & L_{a_{42}}
\end{array}\right]$$

$$M.M^T = \left[\begin{array}{c|cc|ccc}
L_{a_1} & L_{a_8} & L_{a_{15}} & L_{a_{22}} & L_{a_{29}} & L_{a_{36}} \\
L_{a_2} & L_{a_9} & L_{a_{16}} & L_{a_{23}} & L_{a_{30}} & L_{a_{37}} \\ \hline
L_{a_3} & L_{a_{10}} & L_{a_{17}} & L_{a_{24}} & L_{a_{31}} & L_{a_{38}} \\
L_{a_4} & L_{a_{11}} & L_{a_{18}} & L_{a_{25}} & L_{a_{32}} & L_{a_{39}} \\
L_{a_5} & L_{a_{12}} & L_{a_{19}} & L_{a_{26}} & L_{a_{33}} & L_{a_{40}} \\
L_{a_6} & L_{a_{13}} & L_{a_{20}} & L_{a_{27}} & L_{a_{34}} & L_{a_{41}} \\ \hline
L_{a_7} & L_{a_{14}} & L_{a_{21}} & L_{a_{28}} & L_{a_{35}} & L_{a_{42}}
\end{array}\right] \times$$

$$\left[\begin{array}{cc|cccc|c}
L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} \\ \hline
L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} \\
L_{a_{15}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \\ \hline
L_{a_{22}} & L_{a_{23}} & L_{a_{24}} & L_{a_{25}} & L_{a_{26}} & L_{a_{27}} & L_{a_{28}} \\
L_{a_{29}} & L_{a_{30}} & L_{a_{31}} & L_{a_{32}} & L_{a_{33}} & L_{a_{34}} & L_{a_{35}} \\
L_{a_{36}} & L_{a_{37}} & L_{a_{38}} & L_{a_{39}} & L_{a_{40}} & L_{a_{41}} & L_{a_{42}}
\end{array}\right]$$

$$
= \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \end{array}\right]
\left[\begin{array}{cc|cccc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} \end{array}\right] +
$$

$$
\left[\begin{array}{cc} L_{a_8} & L_{a_{15}} \\ L_{a_9} & L_{a_{16}} \\ \hline L_{a_{10}} & L_{a_{17}} \\ L_{a_{11}} & L_{a_{18}} \\ L_{a_{12}} & L_{a_{19}} \\ L_{a_{13}} & L_{a_{20}} \\ \hline L_{a_{14}} & L_{a_{21}} \end{array}\right]
\left[\begin{array}{cc|cccc|c} L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} \\ L_{a_{15}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \end{array}\right] +
$$

$$
\left[\begin{array}{ccc} L_{a_{22}} & L_{a_{29}} & L_{a_{36}} \\ L_{a_{23}} & L_{a_{30}} & L_{a_{37}} \\ \hline L_{a_{24}} & L_{a_{31}} & L_{a_{38}} \\ L_{a_{25}} & L_{a_{32}} & L_{a_{39}} \\ L_{a_{26}} & L_{a_{33}} & L_{a_{40}} \\ L_{a_{27}} & L_{a_{34}} & L_{a_{41}} \\ \hline L_{a_{28}} & L_{a_{35}} & L_{a_{42}} \end{array}\right]
\left[\begin{array}{cc|cccc|c} L_{a_{22}} & L_{a_{23}} & L_{a_{24}} & L_{a_{25}} & L_{a_{26}} & L_{a_{27}} & L_{a_{28}} \\ L_{a_{29}} & L_{a_{30}} & L_{a_{31}} & L_{a_{32}} & L_{a_{33}} & L_{a_{34}} & L_{a_{35}} \\ L_{a_{36}} & L_{a_{37}} & L_{a_{38}} & L_{a_{39}} & L_{a_{40}} & L_{a_{41}} & L_{a_{42}} \end{array}\right]
$$

$$
=\left[\begin{array}{c|c|c}
\begin{bmatrix} L_{a_{22}} & L_{a_{29}} & L_{a_{36}} \\ L_{a_{23}} & L_{a_{30}} & L_{a_{37}} \end{bmatrix}\begin{bmatrix} L_{a_{22}} & L_{a_{23}} \\ L_{a_{29}} & L_{a_{30}} \\ L_{a_{36}} & L_{a_{37}} \end{bmatrix} &
\begin{bmatrix} L_{a_{22}} & L_{a_{29}} & L_{a_{36}} \\ L_{a_{23}} & L_{a_{30}} & L_{a_{37}} \end{bmatrix}\begin{bmatrix} L_{a_{24}} & L_{a_{25}} & L_{a_{26}} & L_{a_{27}} \\ L_{a_{31}} & L_{a_{32}} & L_{a_{33}} & L_{a_{34}} \\ L_{a_{38}} & L_{a_{39}} & L_{a_{40}} & L_{a_{41}} \end{bmatrix} &
\begin{bmatrix} L_{a_{22}} & L_{a_{29}} & L_{a_{36}} \\ L_{a_{23}} & L_{a_{30}} & L_{a_{37}} \end{bmatrix}\begin{bmatrix} L_{a_{28}} \\ L_{a_{35}} \\ L_{a_{42}} \end{bmatrix} \\ \hline
\begin{bmatrix} L_{a_{24}} & L_{a_{31}} & L_{a_{38}} \\ L_{a_{25}} & L_{a_{32}} & L_{a_{39}} \\ L_{a_{26}} & L_{a_{33}} & L_{a_{40}} \\ L_{a_{27}} & L_{a_{34}} & L_{a_{41}} \end{bmatrix}\begin{bmatrix} L_{a_{22}} & L_{a_{23}} \\ L_{a_{29}} & L_{a_{30}} \\ L_{a_{36}} & L_{a_{37}} \end{bmatrix} &
\begin{bmatrix} L_{a_{24}} & L_{a_{31}} & L_{a_{38}} \\ L_{a_{25}} & L_{a_{32}} & L_{a_{39}} \\ L_{a_{26}} & L_{a_{33}} & L_{a_{40}} \\ L_{a_{27}} & L_{a_{34}} & L_{a_{41}} \end{bmatrix}\begin{bmatrix} L_{a_{24}} & L_{a_{25}} & L_{a_{26}} & L_{a_{27}} \\ L_{a_{31}} & L_{a_{32}} & L_{a_{33}} & L_{a_{34}} \\ L_{a_{38}} & L_{a_{39}} & L_{a_{40}} & L_{a_{41}} \end{bmatrix} &
\begin{bmatrix} L_{a_{24}} & L_{a_{31}} & L_{a_{38}} \\ L_{a_{25}} & L_{a_{32}} & L_{a_{39}} \\ L_{a_{26}} & L_{a_{33}} & L_{a_{40}} \\ L_{a_{27}} & L_{a_{34}} & L_{a_{41}} \end{bmatrix}\begin{bmatrix} L_{a_{28}} \\ L_{a_{35}} \\ L_{a_{42}} \end{bmatrix} \\ \hline
\begin{bmatrix} L_{a_{28}} & L_{a_{29}} & L_{a_{30}} \end{bmatrix}\begin{bmatrix} L_{a_{22}} & L_{a_{23}} \\ L_{a_{29}} & L_{a_{30}} \\ L_{a_{36}} & L_{a_{37}} \end{bmatrix} &
\begin{bmatrix} L_{a_{28}} & L_{a_{29}} & L_{a_{30}} \end{bmatrix}\begin{bmatrix} L_{a_{24}} & L_{a_{25}} & L_{a_{26}} & L_{a_{27}} \\ L_{a_{31}} & L_{a_{32}} & L_{a_{33}} & L_{a_{34}} \\ L_{a_{38}} & L_{a_{39}} & L_{a_{40}} & L_{a_{41}} \end{bmatrix} &
\begin{bmatrix} L_{a_{28}} & L_{a_{29}} & L_{a_{30}} \end{bmatrix}\begin{bmatrix} L_{a_{28}} \\ L_{a_{35}} \\ L_{a_{42}} \end{bmatrix}
\end{array}\right]
$$

$$
+\left[\begin{array}{c|c|c}
\begin{bmatrix} L_{a_1} \\ L_{a_2} \end{bmatrix}\begin{pmatrix} L_{a_1} & L_{a_2} \end{pmatrix} & \begin{bmatrix} L_{a_1} \\ L_{a_2} \end{bmatrix}\begin{pmatrix} L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} \end{pmatrix} & \begin{bmatrix} L_{a_1} \\ L_{a_2} \end{bmatrix} L_{a_7} \\
\hline
\begin{bmatrix} L_{a_3} \\ L_{a_4} \\ L_{a_5} \\ L_{a_6} \end{bmatrix}\begin{pmatrix} L_{a_1} & L_{a_2} \end{pmatrix} & \begin{bmatrix} L_{a_3} \\ L_{a_4} \\ L_{a_5} \\ L_{a_6} \end{bmatrix}\begin{pmatrix} L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} \end{pmatrix} & \begin{bmatrix} L_{a_3} \\ L_{a_4} \\ L_{a_5} \\ L_{a_6} \end{bmatrix} L_{a_7} \\
\hline
L_{a_7}\begin{pmatrix} L_{a_1} & L_{a_2} \end{pmatrix} & L_{a_7}\begin{pmatrix} L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} \end{pmatrix} & L_{a_7}.L_{a_7}
\end{array}\right]
$$

$$
+\left[\begin{array}{c|c|c}
\begin{bmatrix} L_{a_8} & L_{a_{15}} \\ L_{a_9} & L_{a_{16}} \end{bmatrix}\begin{bmatrix} L_{a_8} & L_{a_9} \\ L_{a_{15}} & L_{a_{16}} \end{bmatrix} & \begin{bmatrix} L_{a_8} & L_{a_{15}} \\ L_{a_9} & L_{a_{16}} \end{bmatrix}\begin{bmatrix} L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} \\ L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \end{bmatrix} & \begin{bmatrix} L_{a_8} & L_{a_{15}} \\ L_{a_9} & L_{a_{16}} \end{bmatrix}\begin{bmatrix} L_{a_{14}} \\ L_{a_{21}} \end{bmatrix} \\
\hline
\begin{bmatrix} L_{a_{10}} & L_{a_{17}} \\ L_{a_{11}} & L_{a_{18}} \\ L_{a_{12}} & L_{a_{19}} \\ L_{a_{13}} & L_{a_{20}} \end{bmatrix}\begin{bmatrix} L_{a_8} & L_{a_9} \\ L_{a_{15}} & L_{a_{16}} \end{bmatrix} & \begin{bmatrix} L_{a_{10}} & L_{a_{17}} \\ L_{a_{11}} & L_{a_{18}} \\ L_{a_{12}} & L_{a_{19}} \\ L_{a_{13}} & L_{a_{20}} \end{bmatrix}\begin{bmatrix} L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} \\ L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \end{bmatrix} & \begin{bmatrix} L_{a_{10}} & L_{a_{17}} \\ L_{a_{11}} & L_{a_{18}} \\ L_{a_{12}} & L_{a_{19}} \\ L_{a_{13}} & L_{a_{20}} \end{bmatrix}\begin{bmatrix} L_{a_{14}} \\ L_{a_{21}} \end{bmatrix} \\
\hline
\begin{bmatrix} L_{a_{14}} & L_{a_{21}} \end{bmatrix}\begin{bmatrix} L_{a_8} & L_{a_9} \\ L_{a_{15}} & L_{a_{16}} \end{bmatrix} & \begin{bmatrix} L_{a_{14}} & L_{a_{21}} \end{bmatrix}\begin{bmatrix} L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} \\ L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \end{bmatrix} & \begin{bmatrix} L_{a_{14}} & L_{a_{21}} \end{bmatrix}\begin{bmatrix} L_{a_{14}} \\ L_{a_{21}} \end{bmatrix}
\end{array}\right]
$$

$$
=\left[\begin{array}{cc|cccc|c}
L_{a_1}L_{a_1} & L_{a_1}L_{a_2} & L_{a_1}L_{a_3} & L_{a_1}L_{a_4} & L_{a_1}L_{a_5} & L_{a_1}L_{a_6} & L_{a_1}L_{a_7} \\
L_{a_2}L_{a_1} & L_{a_2}L_{a_2} & L_{a_2}L_{a_3} & L_{a_2}L_{a_4} & L_{a_2}L_{a_5} & L_{a_2}L_{a_6} & L_{a_2}L_{a_7} \\
\hline
L_{a_3}L_{a_1} & L_{a_3}L_{a_2} & L_{a_3}L_{a_3} & L_{a_3}L_{a_4} & L_{a_3}L_{a_5} & L_{a_3}L_{a_6} & L_{a_3}L_{a_7} \\
L_{a_4}L_{a_1} & L_{a_4}L_{a_2} & L_{a_4}L_{a_3} & L_{a_4}L_{a_4} & L_{a_4}L_{a_5} & L_{a_4}L_{a_6} & L_{a_4}L_{a_7} \\
L_{a_5}L_{a_1} & L_{a_5}L_{a_2} & L_{a_5}L_{a_3} & L_{a_5}L_{a_4} & L_{a_5}L_{a_5} & L_{a_5}L_{a_6} & L_{a_5}L_{a_7} \\
L_{a_6}L_{a_1} & L_{a_6}L_{a_2} & L_{a_6}L_{a_3} & L_{a_6}L_{a_4} & L_{a_6}L_{a_5} & L_{a_6}L_{a_6} & L_{a_6}L_{a_7} \\
\hline
L_{a_7}L_{a_1} & L_{a_7}L_{a_2} & L_{a_7}L_{a_3} & L_{a_7}L_{a_4} & L_{a_7}L_{a_5} & L_{a_7}L_{a_6} & L_{a_7}L_{a_7}
\end{array}\right]
$$

$$
+\left[\begin{array}{cc|cc}
L_{a_8}L_{a_8}+L_{a_{15}}L_{a_{15}} & L_{a_8}L_{a_9}+L_{a_{15}}L_{a_{16}} & L_{a_8}L_{a_{10}}+L_{a_{15}}L_{a_{17}} & L_{a_8}L_{a_{11}}+L_{a_{15}}L_{a_{18}} \\
L_{a_9}L_{a_8}+L_{a_{16}}L_{a_{15}} & L_{a_9}L_{a_9}+L_{a_{16}}L_{a_{16}} & L_{a_9}L_{a_{10}}+L_{a_{16}}L_{a_{17}} & L_{a_9}L_{a_{11}}+L_{a_{16}}L_{a_{18}} \\
\hline
L_{a_{10}}L_{a_8}+L_{a_{17}}L_{a_{15}} & L_{a_{10}}L_{a_9}+L_{a_{17}}L_{a_{16}} & L_{a_{10}}L_{a_{10}}+L_{a_{17}}L_{a_{17}} & L_{a_{10}}L_{a_{11}}+L_{a_{17}}L_{a_{18}} \\
L_{a_{11}}L_{a_8}+L_{a_{18}}L_{a_{15}} & L_{a_{11}}L_{a_9}+L_{a_{18}}L_{a_{16}} & L_{a_{11}}L_{a_{10}}+L_{a_{18}}L_{a_{17}} & L_{a_{11}}L_{a_{11}}+L_{a_{18}}L_{a_{18}} \\
L_{a_{12}}L_{a_8}+L_{a_{19}}L_{a_{15}} & L_{a_{12}}L_{a_9}+L_{a_{19}}L_{a_{16}} & L_{a_{12}}L_{a_{10}}+L_{a_{19}}L_{a_{17}} & L_{a_{12}}L_{a_{11}}+L_{a_{19}}L_{a_{18}} \\
L_{a_{13}}L_{a_8}+L_{a_{20}}L_{a_{15}} & L_{a_{13}}L_{a_9}+L_{a_{20}}L_{a_{16}} & L_{a_{13}}L_{a_{10}}+L_{a_{20}}L_{a_{17}} & L_{a_{13}}L_{a_{11}}+L_{a_{20}}L_{a_{18}} \\
\hline
L_{a_{14}}L_{a_8}+L_{a_{21}}L_{a_{15}} & L_{a_{14}}L_{a_9}+L_{a_{21}}L_{a_{16}} & L_{a_{14}}L_{a_{10}}+L_{a_{21}}L_{a_{17}} & L_{a_{14}}L_{a_{11}}+L_{a_{21}}L_{a_{18}}
\end{array}\right.
$$

$$
\left.\begin{array}{cc|c}
L_{a_9}L_{a_{12}}+L_{a_{16}}L_{a_{19}} & L_{a_8}L_{a_{13}}+L_{a_{15}}L_{a_{20}} & L_{a_8}L_{a_{14}}+L_{a_{15}}L_{a_{21}} \\
L_{a_9}L_{a_{12}}+L_{a_{16}}L_{a_{19}} & L_{a_9}L_{a_{13}}+L_{a_{16}}L_{a_{20}} & L_{a_9}L_{a_{14}}+L_{a_{16}}L_{a_{21}} \\
\hline
L_{a_{10}}L_{a_{12}}+L_{a_{17}}L_{a_{19}} & L_{a_{10}}L_{a_{13}}+L_{a_{17}}L_{a_{20}} & L_{a_{10}}L_{a_{14}}+L_{a_{17}}L_{a_{21}} \\
L_{a_{11}}L_{a_{12}}+L_{a_{18}}L_{a_{19}} & L_{a_{11}}L_{a_{13}}+L_{a_{18}}L_{a_{20}} & L_{a_{11}}L_{a_{14}}+L_{a_{18}}L_{a_{21}} \\
L_{a_{12}}L_{a_{12}}+L_{a_{19}}L_{a_{19}} & L_{a_{12}}L_{a_{13}}+L_{a_{19}}L_{a_{20}} & L_{a_{12}}L_{a_{14}}+L_{a_{19}}L_{a_{21}} \\
L_{a_{13}}L_{a_{12}}+L_{a_{20}}L_{a_{19}} & L_{a_{13}}L_{a_{13}}+L_{a_{20}}L_{a_{20}} & L_{a_{13}}L_{a_{14}}+L_{a_{20}}L_{a_{21}} \\
\hline
L_{a_{14}}L_{a_{12}}+L_{a_{21}}L_{a_{19}} & L_{a_{14}}L_{a_{13}}+L_{a_{21}}L_{a_{20}} & L_{a_{14}}L_{a_{14}}+L_{a_{21}}L_{a_{21}}
\end{array}\right] +
$$

$$
\left[\begin{array}{cc|c}
L_{a_{22}}L_{a_{22}}+L_{a_{29}}L_{a_{29}}+L_{a_{36}}L_{a_{36}} & L_{a_{22}}L_{a_{23}}+L_{a_{29}}L_{a_{30}}+L_{a_{36}}L_{a_{37}} & L_{a_{22}}L_{a_{24}}+L_{a_{29}}L_{a_{31}}+L_{a_{36}}L_{a_{38}} \\
L_{a_{23}}L_{a_{22}}+L_{a_{30}}L_{a_{29}}+L_{a_{37}}L_{a_{36}} & L_{a_{23}}L_{a_{23}}+L_{a_{30}}L_{a_{30}}+L_{a_{37}}L_{a_{37}} & L_{a_{23}}L_{a_{25}}+L_{a_{30}}L_{a_{31}}+L_{a_{37}}L_{a_{38}} \\
\hline
L_{a_{24}}L_{a_{22}}+L_{a_{31}}L_{a_{29}}+L_{a_{38}}L_{a_{36}} & L_{a_{24}}L_{a_{23}}+L_{a_{31}}L_{a_{30}}+L_{a_{38}}L_{a_{37}} & L_{a_{24}}L_{a_{24}}+L_{a_{31}}L_{a_{31}}+L_{a_{38}}L_{a_{38}} \\
L_{a_{25}}L_{a_{22}}+L_{a_{32}}L_{a_{29}}+L_{a_{39}}L_{a_{36}} & L_{a_{25}}L_{a_{23}}+L_{a_{32}}L_{a_{30}}+L_{a_{39}}L_{a_{37}} & L_{a_{25}}L_{a_{24}}+L_{a_{32}}L_{a_{31}}+L_{a_{39}}L_{a_{38}} \\
L_{a_{26}}L_{a_{22}}+L_{a_{33}}L_{a_{29}}+L_{a_{40}}L_{a_{36}} & L_{a_{26}}L_{a_{23}}+L_{a_{33}}L_{a_{30}}+L_{a_{40}}L_{a_{37}} & L_{a_{26}}L_{a_{24}}+L_{a_{33}}L_{a_{31}}+L_{a_{40}}L_{a_{38}} \\
L_{a_{27}}L_{a_{22}}+L_{a_{34}}L_{a_{29}}+L_{a_{41}}L_{a_{36}} & L_{a_{27}}L_{a_{23}}+L_{a_{34}}L_{a_{30}}+L_{a_{41}}L_{a_{37}} & L_{a_{27}}L_{a_{24}}+L_{a_{34}}L_{a_{31}}+L_{a_{41}}L_{a_{38}} \\
\hline
L_{a_{28}}L_{a_{22}}+L_{a_{35}}L_{a_{29}}+L_{a_{42}}L_{a_{36}} & L_{a_{28}}L_{a_{23}}+L_{a_{35}}L_{a_{30}}+L_{a_{42}}L_{a_{37}} & L_{a_{28}}L_{a_{24}}+L_{a_{35}}L_{a_{31}}+L_{a_{42}}L_{a_{38}}
\end{array}\right.
$$

$$
\begin{array}{ccc}
L_{a_{22}}L_{a_{25}}+L_{a_{29}}L_{a_{32}}+L_{a_{36}}L_{a_{39}} & L_{a_{22}}L_{a_{26}}+L_{a_{29}}L_{a_{33}}+L_{a_{36}}L_{a_{40}} & L_{a_{22}}L_{a_{27}}+L_{a_{29}}L_{a_{34}}+L_{a_{36}}L_{a_{41}} \\
L_{a_{23}}L_{a_{25}}+L_{a_{30}}L_{a_{32}}+L_{a_{37}}L_{a_{39}} & L_{a_{23}}L_{a_{26}}+L_{a_{30}}L_{a_{33}}+L_{a_{37}}L_{a_{40}} & L_{a_{23}}L_{a_{27}}+L_{a_{30}}L_{a_{34}}+L_{a_{37}}L_{a_{41}} \\
\hline
L_{a_{24}}L_{a_{25}}+L_{a_{31}}L_{a_{32}}+L_{a_{38}}L_{a_{39}} & L_{a_{24}}L_{a_{26}}+L_{a_{31}}L_{a_{33}}+L_{a_{38}}L_{a_{40}} & L_{a_{24}}L_{a_{27}}+L_{a_{31}}L_{a_{34}}+L_{a_{38}}L_{a_{41}} \\
L_{a_{25}}L_{a_{25}}+L_{a_{32}}L_{a_{32}}+L_{a_{39}}L_{a_{39}} & L_{a_{25}}L_{a_{26}}+L_{a_{32}}L_{a_{33}}+L_{a_{39}}L_{a_{40}} & L_{a_{25}}L_{a_{27}}+L_{a_{32}}L_{a_{34}}+L_{a_{39}}L_{a_{41}} \\
L_{a_{26}}L_{a_{25}}+L_{a_{33}}L_{a_{32}}+L_{a_{40}}L_{a_{39}} & L_{a_{26}}L_{a_{26}}+L_{a_{33}}L_{a_{33}}+L_{a_{40}}L_{a_{40}} & L_{a_{26}}L_{a_{27}}+L_{a_{33}}L_{a_{34}}+L_{a_{40}}L_{a_{41}} \\
L_{a_{27}}L_{a_{25}}+L_{a_{34}}L_{a_{32}}+L_{a_{41}}L_{a_{39}} & L_{a_{27}}L_{a_{26}}+L_{a_{34}}L_{a_{33}}+L_{a_{41}}L_{a_{40}} & L_{a_{27}}L_{a_{27}}+L_{a_{34}}L_{a_{34}}+L_{a_{41}}L_{a_{41}} \\
\hline
L_{a_{28}}L_{a_{25}}+L_{a_{35}}L_{a_{32}}+L_{a_{42}}L_{a_{39}} & L_{a_{28}}L_{a_{26}}+L_{a_{35}}L_{a_{33}}+L_{a_{42}}L_{a_{40}} & L_{a_{28}}L_{a_{27}}+L_{a_{35}}L_{a_{34}}+L_{a_{42}}L_{a_{41}}
\end{array}
$$

$$
\left.\begin{array}{|c}
L_{a_{22}}L_{a_{28}}+L_{a_{29}}L_{a_{35}}+L_{a_{36}}L_{a_{42}} \\
L_{a_{23}}L_{a_{28}}+L_{a_{30}}L_{a_{35}}+L_{a_{37}}L_{a_{42}} \\
\hline
L_{a_{24}}L_{a_{28}}+L_{a_{31}}L_{a_{35}}+L_{a_{38}}L_{a_{42}} \\
L_{a_{25}}L_{a_{28}}+L_{a_{32}}L_{a_{35}}+L_{a_{39}}L_{a_{42}} \\
L_{a_{26}}L_{a_{28}}+L_{a_{33}}L_{a_{35}}+L_{a_{40}}L_{a_{42}} \\
L_{a_{27}}L_{a_{28}}+L_{a_{34}}L_{a_{35}}+L_{a_{41}}L_{a_{42}} \\
\hline
L_{a_{28}}L_{a_{28}}+L_{a_{35}}L_{a_{35}}+L_{a_{42}}L_{a_{42}}
\end{array}\right]
$$

$$
=\left[\begin{array}{cc|cccc|c}
L_{a_1^2/m+1} & L_{a_1a_2/m+1} & L_{a_1a_3/m+1} & L_{a_1a_4/m+1} & L_{a_1a_5/m+1} & L_{a_1a_6/m+1} & L_{a_1a_7/m+1} \\
L_{a_2a_1/m+1} & L_{a_2^2/m+1} & L_{a_2a_3/m+1} & L_{a_2a_4/m+1} & L_{a_2a_5/m+1} & L_{a_2a_6/m+1} & L_{a_2a_7/m+1} \\
\hline
L_{a_3a_1/m+1} & L_{a_3a_2/m+1} & L_{a_3^2/m+1} & L_{a_3a_4/m+1} & L_{a_3a_5/m+1} & L_{a_3a_6/m+1} & L_{a_3a_7/m+1} \\
L_{a_4a_1/m+1} & L_{a_4a_2/m+1} & L_{a_4a_3/m+1} & L_{a_4^2/m+1} & L_{a_4a_5/m+1} & L_{a_4a_6/m+1} & L_{a_4a_7/m+1} \\
L_{a_5a_1/m+1} & L_{a_5a_2/m+1} & L_{a_5a_3/m+1} & L_{a_5a_4/m+1} & L_{a_5^2/m+1} & L_{a_5a_6/m+1} & L_{a_5a_7/m+1} \\
L_{a_6a_1/m+1} & L_{a_6a_2/m+1} & L_{a_6a_3/m+1} & L_{a_6a_4/m+1} & L_{a_6a_5/m+1} & L_{a_6^2/m+1} & L_{a_6a_7/m+1} \\
\hline
L_{a_7a_1/m+1} & L_{a_7a_2/m+1} & L_{a_7a_3/m+1} & L_{a_7a_4/m+1} & L_{a_7a_5/m+1} & L_{a_7a_6/m+1} & L_{a_7^2/m+1}
\end{array}\right]
$$

$$
+\left[\begin{array}{cc|cc}
L_{a_8^2+a_{15}^2/m+1} & L_{a_8 a_9+a_{15} a_{16}/m+1} & L_{a_8 a_{10}+a_{15} a_{17}/m+1} & L_{a_8 a_{11}+a_{15} a_{18}/m+1} \\
L_{a_9 a_8+a_{16} a_{15}/m+1} & L_{a_9^2+a_{16}^2/m+1} & L_{a_9 a_{10}+a_{16} a_{17}/m+1} & L_{a_9 a_{11}+a_{16} a_{18}/m+1} \\
\hline
L_{a_{10} a_8+a_{17} a_{15}/m+1} & L_{a_{10} a_9+a_{17} a_{16}/m+1} & L_{a_{10}^2+a_{17}^2/m+1} & L_{a_{10} a_{11}+a_{17} a_{18}/m+1} \\
L_{a_{11} a_8+a_{18} a_{15}/m+1} & L_{a_{11} a_9+a_{18} a_{16}/m+1} & L_{a_{11} a_{10}+a_{18} a_{17}/m+1} & L_{a_{11}^2+a_{18}^2/m+1} \\
L_{a_{12} a_8+a_{19} a_{15}/m+1} & L_{a_{12} a_9+a_{19} a_{16}/m+1} & L_{a_{12} a_{10}+a_{19} a_{17}/m+1} & L_{a_{12} a_{11}+a_{19} a_{18}/m+1} \\
L_{a_{13} a_8+a_{20} a_{15}/m+1} & L_{a_{13} a_9+a_{20} a_{16}/m+1} & L_{a_{13} a_{10}+a_{20} a_{17}/m+1} & L_{a_{13} a_{11}+a_{20} a_{18}/m+1} \\
\hline
L_{a_{14} a_8+a_{21} a_{15}/m+1} & L_{a_{14} a_9+a_{21} a_{16}/m+1} & L_{a_{14} a_{10}+a_{21} a_{17}/m+1} & L_{a_{14} a_{11}+a_{21} a_{18}/m+1}
\end{array}\right.
$$

$$
\left.\begin{array}{cc|c}
L_{a_8 a_{12}+a_{15} a_{19}/m+1} & L_{a_8 a_{13}+a_{15} a_{20}/m+1} & L_{a_8 a_{14}+a_{15} a_{21}/m+1} \\
L_{a_9 a_{12}+a_{16} a_{19}/m+1} & L_{a_9 a_{13}+a_{16} a_{20}/m+1} & L_{a_9 a_{14}+a_{16} a_{21}/m+1} \\
\hline
L_{a_{10} a_{12}+a_{17} a_{19}/m+1} & L_{a_{10} a_{13}+a_{17} a_{20}/m+1} & L_{a_{10} a_{14}+a_{17} a_{21}/m+1} \\
L_{a_{11} a_{12}+a_{18} a_{19}/m+1} & L_{a_{11} a_{13}+a_{18} a_{20}/m+1} & L_{a_{11} a_{14}+a_{18} a_{21}/m+1} \\
L_{a_{12}^2+a_{19}^2/m+1} & L_{a_{12} a_{13}+a_{19} a_{20}/m+1} & L_{a_{12} a_{14}+a_{19} a_{21}/m+1} \\
L_{a_{13} a_{12}+a_{20} a_{19}/m+1} & L_{a_{13}^2+a_{20}^2/m+1} & L_{a_{13} a_{14}+a_{20} a_{21}/m+1} \\
\hline
L_{a_{14} a_{12}+a_{21} a_{19}/m+1} & L_{a_{14} a_{13}+a_{21} a_{20}/m+1} & L_{a_{14}^2+a_{21}^2/m+1}
\end{array}\right] +
$$

$$
\left[\begin{array}{cc|c}
L_{a_{22}^2+a_{29}^2+a_{36}^2/m+1} & L_{a_{22} a_{23}+a_{29} a_{30}+a_{36} a_{37}/m+1} & L_{a_{22} a_{24}+a_{29} a_{31}+a_{36} a_{38}/m+1} \\
L_{a_{23} a_{22}+a_{30} a_{29}+a_{37} a_{36}/m+1} & L_{a_{23}^2+a_{30}^2+a_{37}^2/m+1} & L_{a_{23} a_{25}+a_{30} a_{31}+a_{37} a_{38}/m+1} \\
\hline
L_{a_{24} a_{22}+a_{31} a_{29}+a_{38} a_{36}/m+1} & L_{a_{24} a_{23}+a_{31} a_{30}+a_{38} a_{37}/m+1} & L_{a_{24}^2+a_{31}^2+a_{38}^2/m+1} \\
L_{a_{25} a_{22}+a_{32} a_{29}+a_{39} a_{36}/m+1} & L_{a_{25} a_{23}+a_{32} a_{30}+a_{39} a_{37}/m+1} & L_{a_{25} a_{24}+a_{32} a_{31}+a_{39} a_{38}/m+1} \\
L_{a_{26} a_{22}+a_{33} a_{29}+a_{40} a_{36}/m+1} & L_{a_{26} a_{23}+a_{33} a_{30}+a_{40} a_{37}/m+1} & L_{a_{26} a_{24}+a_{33} a_{31}+a_{40} a_{38}/m+1} \\
L_{a_{27} a_{22}+a_{34} a_{29}+a_{41} a_{36}/m+1} & L_{a_{27} a_{23}+a_{34} a_{30}+a_{41} a_{37}/m+1} & L_{a_{27} a_{24}+a_{34} a_{31}+a_{41} a_{38}/m+1} \\
\hline
L_{a_{28} a_{22}+a_{35} a_{29}+a_{42} a_{36}/m+1} & L_{a_{28} a_{23}+a_{35} a_{30}+a_{42} a_{37}/m+1} & L_{a_{28} a_{24}+a_{35} a_{31}+a_{42} a_{38}/m+1}
\end{array}\right.
$$

$$
\begin{array}{ccc}
L_{a_{22} a_{25}+a_{29} a_{32}+a_{36} a_{39} / m+1} & L_{a_{22} a_{26}+a_{29} a_{33}+a_{36} a_{40} / m+1} & L_{a_{22} a_{27}+a_{29} a_{34}+a_{36} a_{41} / m+1} \\
L_{a_{23} a_{25}+a_{30} a_{32}+a_{37} a_{39} / m+1} & L_{a_{23} a_{26}+a_{30} a_{33}+a_{37} a_{40} / m+1} & L_{a_{23} a_{27}+a_{30} a_{34}+a_{37} a_{41} / m+1} \\
\hline
L_{a_{24} a_{25}+a_{31} a_{32}+a_{38} a_{39} / m+1} & L_{a_{24} a_{26}+a_{31} a_{33}+a_{38} a_{40} / m+1} & L_{a_{24} a_{27}+a_{31} a_{34}+a_{38} a_{41} / m+1} \\
L_{a_{25}^2+a_{32}^2+a_{41}^2 / m+1} & L_{a_{25} a_{26}+a_{32} a_{33}+a_{39} a_{40} / m+1} & L_{a_{25} a_{27}+a_{32} a_{34}+a_{39} a_{41} / m+1} \\
L_{a_{26} a_{25}+a_{33} a_{32}+a_{40} a_{39} / m+1} & L_{a_{26}^2+a_{33}^2+a_{40}^2 / m+1} & L_{a_{26} a_{27}+a_{33} a_{34}+a_{40} a_{41} / m+1} \\
L_{a_{27} a_{25}+a_{34} a_{32}+a_{41} a_{39} / m+1} & L_{a_{27}} L_{a_{26}+a_{34} a_{33}+a_{41} a_{40} / m+1} & L_{a_{27}^2+a_{34}^2+a_{41}^2 / m+1} \\
\hline
L_{a_{28} a_{25}+a_{35} a_{32}+a_{42} a_{39} / m+1} & L_{a_{28} a_{26}+a_{35} a_{33}+a_{42} a_{40} / m+1} & L_{a_{28} a_{27}+a_{35} a_{34}+a_{42} a_{41} / m+1}
\end{array}
$$

$$
\left.
\begin{array}{|c}
L_{a_{22} a_{28}+a_{29} a_{35}+a_{36} a_{42} / m+1} \\
L_{a_{23} a_{28}+a_{30} a_{35}+a_{37} a_{42} / m+1} \\
\hline
L_{a_{24} a_{28}+a_{31} a_{35}+a_{38} a_{42} / m+1} \\
L_{a_{25} a_{28}+a_{32} a_{35}+a_{39} a_{42} / m+1} \\
L_{a_{26} a_{28}+a_{33} a_{35}+a_{40} a_{42} / m+1} \\
L_{a_{27} a_{28}+a_{34} a_{35}+a_{41} a_{42} / m+1} \\
\hline
L_{a_{28}^2+a_{35}^2+a_{42}^2 / m+1}
\end{array}
\right]
$$

$$
= \left[
\begin{array}{cc}
L_{a_1^2+a_8^2+a_{15}^2+a_{22}^2+a_{29}^2+a_{36}^2 / m+1} & L_{a_1 a_2+a_8 a_9+a_{15} a_{16}+a_{22} a_{23}+a_{29} a_{30}+a_{36} a_{37} / m+1} \\
L_{a_2 a_1+a_9 a_8+a_{16} a_{15}+a_{23} a_{22}+a_{30} a_{29}+a_{37} a_{36} / m+1} & L_{a_2^2+a_9^2+a_{16}^2+a_{23}^2+a_{30}^2+a_{37}^2 / m+1} \\
\hline
L_{a_3 a_1+a_{10} a_8+a_{17} a_{15}+a_{24} a_{22}+a_{31} a_{29}+a_{38} a_{36} / m+1} & L_{a_3 a_2+a_{10} a_9+a_{17} a_{16}+a_{24} a_{23}+a_{31} a_{30}+a_{38} a_{37} / m+1} \\
L_{a_4 a_1+a_{11} a_8+a_{18} a_{15}+a_{25} a_{22}+a_{32} a_{29}+a_{39} a_{36} / m+1} & L_{a_4 a_2+a_{11} a_9+a_{18} a_{16}+a_{25} a_{23}+a_{32} a_{30}+a_{39} a_{37} / m+1} \\
L_{a_5 a_1+a_{12} a_8+a_{19} a_{15}+a_{26} a_{22}+a_{33} a_{29}+a_{40} a_{36} / m+1} & L_{a_5 a_2+a_{12} a_9+a_{19} a_{16}+a_{26} a_{23}+a_{33} a_{30}+a_{40} a_{37} / m+1} \\
L_{a_6 a_1+a_{13} a_8+a_{20} a_{15}+a_{27} a_{22}+a_{34} a_{29}+a_{41} a_{36} / m+1} & L_{a_6 a_2+a_{13} a_9+a_{20} a_{16}+a_{27} a_{23}+a_{34} a_{30}+a_{41} a_{37} / m+1} \\
\hline
L_{a_7 a_1+a_{14} a_8+a_{21} a_{15}+a_{28} a_{22}+a_{35} a_{29}+a_{42} a_{36} / m+1} & L_{a_7 a_2+a_{14} a_9+a_{21} a_{16}+a_{28} a_{23}+a_{35} a_{30}+a_{42} a_{37} / m+1}
\end{array}
\right.
$$

$$
\left.\begin{array}{cc}
L_{a_1a_3+a_8a_{10}+a_{15}a_{17}+a_{22}a_{24}+a_{29}a_{31}+a_{36}a_{38}/m+1} & L_{a_1a_4+a_8a_{11}+a_{15}a_{18}+a_{22}a_{25}+a_{29}a_{32}+a_{36}a_{39}/m+1} \\
L_{a_2a_3+a_9a_{10}+a_{16}a_{17}+a_{23}a_{24}+a_{30}a_{31}+a_{37}a_{38}/m+1} & L_{a_2a_4+a_9a_{11}+a_{16}a_{18}+a_{23}a_{25}+a_{30}a_{32}+a_{37}a_{39}/m+1} \\
\hline
L_{a_3^2+a_{10}^2+a_{17}^2+a_{24}^2+a_{31}^2+a_{38}^2/m+1} & L_{a_3a_4+a_{10}a_{11}+a_{17}a_{18}+a_{24}a_{25}+a_{31}a_{32}+a_{38}a_{39}/m+1} \\
L_{a_4a_3+a_{11}a_{10}+a_{18}a_{17}+a_{25}a_{24}+a_{32}a_{31}+a_{39}a_{38}/m+1} & L_{a_4^2+a_{11}^2+a_{18}^2+a_{25}^2+a_{32}^2+a_{39}^2/m+1} \\
L_{a_5a_3+a_{12}a_{10}+a_{19}a_{17}+a_{26}a_{24}+a_{33}a_{31}+a_{40}a_{38}/m+1} & L_{a_5a_4+a_{12}a_{11}+a_{19}a_{18}+a_{26}a_{25}+a_{33}a_{32}+a_{40}a_{39}/m+1} \\
L_{a_6a_3+a_{13}a_{10}+a_{20}a_{17}+a_{27}a_{24}+a_{34}a_{31}+a_{41}a_{38}/m+1} & L_{a_6a_4+a_{13}a_{11}+a_{20}a_{18}+a_{27}a_{25}+a_{34}a_{32}+a_{41}a_{39}/m+1} \\
\hline
L_{a_7a_3+a_{14}a_{10}+a_{21}a_{17}+a_{28}a_{24}+a_{35}a_{31}+a_{42}a_{38}/m+1} & L_{a_7a_4+a_{14}a_{11}+a_{21}a_{18}+a_{28}a_{25}+a_{35}a_{32}+a_{42}a_{39}/m+1}
\end{array}\right.
$$

$$
\begin{array}{cc}
L_{a_1a_5+a_8a_{12}+a_{15}a_{19}+a_{22}a_{26}+a_{29}a_{33}+a_{36}a_{40}/m+1} & L_{a_1a_6+a_8a_{13}+a_{15}a_{20}+a_{22}a_{27}+a_{29}a_{34}+a_{36}a_{41}/m+1} \\
L_{a_2a_5+a_9a_{12}+a_{16}a_{19}+a_{23}a_{26}+a_{30}a_{33}+a_{37}a_{40}/m+1} & L_{a_2a_6+a_9a_{13}+a_{16}a_{20}+a_{23}a_{27}+a_{30}a_{34}+a_{37}a_{41}/m+1} \\
\hline
L_{a_3a_5+a_{10}a_{12}+a_{17}a_{19}+a_{24}a_{26}+a_{31}a_{33}+a_{38}a_{40}/m+1} & L_{a_3a_6+a_{10}a_{13}+a_{17}a_{20}+a_{24}a_{27}+a_{31}a_{34}+a_{38}a_{41}/m+1} \\
L_{a_4a_5+a_{11}a_{12}+a_{18}a_{19}+a_{25}a_{26}+a_{32}a_{33}+a_{39}a_{40}/m+1} & L_{a_6a_6+a_{11}a_{13}+a_{18}a_{20}+a_{25}a_{27}+a_{32}a_{34}+a_{39}a_{41}/m+1} \\
L_{a_5^2+a_{12}^2+a_{19}^2+a_{26}^2+a_{33}^2+a_{40}^2/m+1} & L_{a_5a_6+a_{12}a_{13}+a_{19}a_{20}+a_{26}a_{27}+a_{33}a_{34}+a_{40}a_{41}/m+1} \\
L_{a_6a_5+a_{13}a_{12}+a_{20}a_{19}+a_{27}a_{26}+a_{34}a_{33}+a_{41}a_{40}/m+1} & L_{a_6^2+a_{13}^2+a_{20}^2+a_{27}^2+a_{34}^2+a_{41}^2/m+1} \\
\hline
L_{a_7a_5+a_{14}a_{12}+a_{21}a_{19}+a_{28}a_{26}+a_{35}a_{33}+a_{42}a_{40}/m+1} & L_{a_7a_6+a_{14}a_{13}+a_{21}a_{20}+a_{28}a_{27}+a_{35}a_{34}+a_{42}a_{41}/m+1}
\end{array}
$$

$$
\left|\begin{array}{c}
L_{a_1a_7+a_8a_{14}+a_{15}a_{21}+a_{22}a_{28}+a_{29}a_{35}+a_{36}a_{42}/m+1} \\
L_{a_2a_7+a_9a_{14}+a_{16}a_{21}+a_{23}a_{28}+a_{30}a_{35}+a_{37}a_{42}/m+1} \\
\hline
L_{a_3a_7+a_{10}a_{14}+a_{17}a_{21}+a_{24}a_{28}+a_{31}a_{35}+a_{38}a_{42}/m+1} \\
L_{a_4a_7+a_{11}a_{14}+a_{18}a_{21}+a_{25}a_{28}+a_{32}a_{35}+a_{39}a_{42}/m+1} \\
L_{a_5a_7+a_{12}a_{14}+a_{19}a_{21}+a_{26}a_{28}+a_{33}a_{35}+a_{40}a_{42}/m+1} \\
L_{a_6a_7+a_{13}a_{14}+a_{20}a_{21}+a_{27}a_{28}+a_{34}a_{35}+a_{41}a_{42}/m+1} \\
\hline
L_{a_7^2+a_{14}^2+a_{21}^2+a_{28}^2+a_{35}^2+a_{42}^2/m+1}
\end{array}\right] .
$$

We see the product is again super matrix of refined labels but is clearly a symmetric super matrix of refined labels of natural order 7 × 7. Thus by this technique of product we can get symmetric super matrix of refined labels which may have applications in real world problems.

We see at times the product of two super matrices of refined labels may turn out to be just matrices or symmetric super matrices of refined labels depending on the matrices producted and the products defined [48].

Now we proceed onto define the notion of M – matrix of labels Z – matrices labels and the corresponding subdirect sums.

Let A = $\left\{ \left(L_{a_i}\right)_{m\times n} \middle| L_{a_i} \in L_R \right\}$ be a m × n refined label matrix we say A > 0 if each $L_{a_i} = \dfrac{a_i}{m+1} > 0$. (A ≥ 0 if each $L_{a_i} = \dfrac{a_i}{m+1} \geq 0$). Thus if A ≥ B we see A – B ≥ 0. Square matrices with refined labels which have non positive off – diagonal entries are called Z-matrices of refined labels.

We see $-L_a = L_{-a} = \dfrac{-a}{m+1}$.

$$A = \begin{bmatrix} L_{a_1} & -L_{a_2} & -L_{a_3} \\ -L_{a_4} & L_{a_5} & -L_{a_6} \\ -L_{a_7} & -L_{a_8} & L_{a_9} \end{bmatrix}$$ is a Z-matrix of refined labels where $a_i > 0$.

We call a Z-matrix of refined labels A to be a non singular M-matrix of refined labels if $A^{-1} \geq 0$.

We have following properties to be true for usual M-matrices which is analogously true in case of M-matrices of refined labels.

(i) The diagonal of a non singular M-matrix of refined labels is positive.

(ii) If B is a Z-matrix of refined labels and M is a non singular M-matrix and M ≤ B then B is also a non singular M-matrix.

In particular any matrix obtained from a non singular M-matrix of refined labels by setting off-diagonal entries is zero is also a non singular M-matrix.
Thus

$$A = \begin{bmatrix} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \end{bmatrix};\ L_{a_1} = \frac{a_1}{m+1} > 0,$$

$$L_{a_5} = \frac{a_5}{m+1} > 0 \text{ and } L_{a_9} = \frac{a_9}{m+1} > 0$$

$$L_{a_2} = \frac{a_2}{m+1} < 0,\ L_{a_3} = \frac{a_3}{m+1} < 0,\ L_{a_4} = \frac{a_4}{m+1} < 0,$$

$$L_{a_6} = \frac{a_6}{m+1} < 0,\ L_{a_7} = \frac{a_7}{m+1} < 0 \text{ and}$$

$$L_{a_8} = \frac{a_8}{m+1} < 0.$$

A is a non singular M-matrix and the diagonal elements are positive.

A matrix M is non singular M-matrix if and only if each principal submatrix of M is a non singular M-matrix.

For instance take

$$A = \left[\begin{array}{c|cc} L_{a_1} & -L_{a_2} & -L_{a_3} \\ \hline -L_{a_4} & L_{a_5} & -L_{a_6} \\ -L_{a_7} & -L_{a_8} & L_{a_9} \end{array}\right]$$

to be a non singular M-matrix then $L_{a_1} \neq 0 \in L_R$ and

$\begin{bmatrix} L_{a_5} & -L_{a_6} \\ -L_{a_8} & L_{a_9} \end{bmatrix}$ are non singular $L_{a_i} > 0$

that is $\frac{a_i}{m+1} > 0.\ 1 \leq i \leq 9$ and $L_{a_i} \in L_R$.

Finally a Z-matrix A is non singular M-matrix if and only if there exists a positive vector $L_x > 0$ such that $AL_x > 0$.

Thus all results enjoyed by simple M-matrices can be derived analogously for M-matrices of refined labels.

Now we can realize these structures as super matrices of refined labels.

Suppose

$$A = \left[\begin{array}{c|cc} L_{a_1} & -L_{a_2} & -L_{a_3} \\ \hline -L_{a_4} & L_{a_5} & -L_{a_6} \\ -L_{a_7} & -L_{a_8} & L_{a_9} \end{array}\right]$$

and

$$B = \left[\begin{array}{cc|c} L_{b_1} & -L_{b_2} & -L_{b_3} \\ -L_{b_4} & L_{b_5} & -L_{b_6} \\ \hline -L_{b_7} & -L_{b_8} & L_{b_9} \end{array}\right]$$

$$L_{b_1} = L_{a_5} \text{ be } -L_{a_6} = -L_{b_2}$$
$$-L_{a_9} = L_{b_5} \text{ and } -L_{b_4} = L_{a_8}.$$

Two super matrices of refined labels then $A \oplus_2 B = C$ is not a M-matrix of refined labels.

$$C = \left[\begin{array}{c|cc|c} L_{a_1} & -L_{a_2} & -L_{a_3} & 0 \\ \hline -L_{a_4} & L_{2b_1} & -L_{2b_2} & -L_{b_3} \\ -L_{a_5} & -L_{2b_4} & L_{2b_5} & -L_{b_6} \\ \hline 0 & -L_{b_7} & -L_{b_8} & L_{b_9} \end{array}\right]$$

We can find C^{-1}. We see C is super matrix of refined labels and the sum is not a usual sum of super matrices.

This method of addition of over lapping super matrices of refined labels may find itself useful in applications. All results studied can be easily defined for matrices of refined labels with simple appropriate operations.

We can also define the P-matrix of refined labels as in case of P-matrix.

Further it is left as an exercise for the reader to prove that in general k-subdirect sum of two P-matrices of refined labels is not a P-matrix of refined labels in general.

If

$$A = \left[\begin{array}{c|cc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & 0 \\ L_{a_6} & 0 & L_{a_7} \end{array}\right] \text{ and } B = \left[\begin{array}{cc|c} L_{b_1} & 0 & L_{b_2} \\ 0 & L_{b_3} & L_{b_4} \\ \hline L_{b_7} & L_{b_6} & L_{b_7} \end{array}\right]$$

be two P-matrices of refined labels then we have the 2-subdirect sum as

$$C = A \oplus_2 B = \left[\begin{array}{c|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & 0 \\ \hline L_{a_4} & L_{a_5+b_1} & 0 & L_{b_2} \\ L_{a_6} & 0 & L_{a_7+b_3} & L_{b_4} \\ \hline 0 & L_{b_5} & L_{b_6} & L_{b_7} \end{array}\right]$$

is not a P-matrix of refined labels as det (C) < 0.

Thus the concept of M-matrices, Z-matrices and P-matrices can also be realized as the super matrices of refined labels. Also the k-subdirect sum is also again a super matrix but addition is different. For in usual super matrices addition can be carried out for which the resultant after addition enjoy the same partition. Here we see the k-subdirect sum is defined only in those cases of different partition and however the k-subdirect sum of super matrices do not enjoy even the same natural order as that of the their components. Thus they do not form a group or a semigroup under k-subdirect addition.

Chapter Four

SUPER VECTOR SPACES USING REFINED LABELS

In this chapter we for the first time introduce the notion of supervector space of refined labels using the super matrices of refined labels. By using the notion of supervector space of refined labels we obtain many vector spaces as against usual matrices of refined labels. We proceed onto illustrate the definitions and properties by examples so that the reader gets a complete grasp of the subject.

DEFINITION 4.1 : *Let V =*

$$\left\{\left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \mid L_{a_4} \quad L_{a_5} \mid \ldots \mid L_{a_r} \quad L_{a_{r+1}} \quad L_{a_{r+2}} \mid \ldots \mid L_{a_n}\right)\right|$$

$L_{a_i} \in L_R; 1 \leq i \leq n \}$ *be the collection of all super row vectors of same type of refined labels. V is an additive abelian group. V is a super row vector space of refined labels over R or* L_R *(as* $L_R \cong R$*).*

We will illustrate this situation by some examples.

Example 4.1: Let

$$V = \left\{\left(L_{a_1} \mid L_{a_2} \quad L_{a_3} \mid L_{a_4} \quad L_{a_5} \quad L_{a_6}\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 6\right\}$$

be a super row vector space of refined labels over the field L_R (or R).

Example 4.2: Let

$V = \left\{\left(L_{a_1} \quad L_{a_2} \mid L_{a_3} \mid L_{a_4} \quad L_{a_5} \quad L_{a_6}\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 6\right\}$ be a super row vector space of refined labels over the field L_R (or R).

Example 4.3: Let

$W = \left\{\left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \quad L_{a_4} \mid L_{a_5} \mid L_{a_6}\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 6\right\}$ be a super row vector space of refined labels over the field L_R (or R).

Example 4.4: Let

$V = \left\{\left(L_{a_1} \quad L_{a_2} \mid L_{a_3} \mid L_{a_4} \mid L_{a_5} \mid L_{a_6}\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 6\right\}$ be a super row vector space of refined labels over the field L_R (or R).

We see for a given row vector $\left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \quad L_{a_4} \quad L_{a_5} \quad L_{a_6}\right)$ of refined labels we can get several super row vector space of refined labels over the field L_R or R. Thus this is one of the main advantage of defining super row vectors of refined labels over L_R or R.

Example 4.5: Let

$X = \{\left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \quad L_{a_4} \mid L_{a_5} \quad L_{a_6} \quad L_{a_7} \mid L_{a_8} \quad L_{a_9} \quad L_{a_{10}}\right) |$

$L_{a_i} \in L_R; 1 \leq i \leq 10$ } be a collection of all super row vectors of refined labels. X is a group under addition and X is a super row vector space of refined labels over L_R (or R).

Now having seen examples of super row vector spaces of refined labels we now proceed onto define substructures in them.

DEFINITION 4.2 : *Let*

$$V = \left\{\left(L_{a_1} \mid L_{a_2} \quad L_{a_3} \mid L_{a_4} \quad L_{a_5} \quad L_{a_6} \quad L_{a_7} \mid \ldots \mid L_{a_{n-1}} \quad L_{a_n}\right)\right|$$

$L_{a_i} \in L_R; 1 \leq i \leq n$ } *be a super row vector space of refined labels over the field L_R. Suppose $W \subseteq V$; W a proper subset of V; if W is a super row vector space of refined labels over L_R. then we define W to be super vector subspace of V over the field L_R.*

Example 4.6: Let

$$P = \left\{\left(L_{a_1} \quad L_{a_2} \mid L_{a_3} \mid L_{a_4} \quad L_{a_5} \quad L_{a_6} \mid L_{a_7}\right)\middle| L_{a_i} \in L_R; 1 \leq i \leq 7\right\}$$

be a super vector space of V over the field L_R (or R).

Consider

$$S = \left\{\left(L_{a_1} \quad 0 \mid L_{a_2} \mid 0 \quad L_{a_3} \quad 0 \mid L_{a_4}\right)\middle| L_{a_i} \in L_R; 1 \leq i \leq 7\right\} \subseteq P$$

is a super row vector subspace of refined labels of P over the field L_R (or R).

Take

$$T = \left\{\left(0 \quad L_{a_1} \mid 0 \mid L_{a_2} \quad 0 \quad L_{a_3} \mid 0\right)\middle| L_{a_i} \in L_R; 1 \leq i \leq 3\right\} \subseteq P,$$

T is a super row vector subspace of refined labels of P.

Example 4.7: Let

$$X = \left\{\left(L_{a_1} \mid L_{a_2} \quad L_{a_3} \quad L_{a_4}\right)\middle| L_{a_i} \in L_R; 1 \leq i \leq 4\right\}$$

be a super row vector space of refined labels over L_R (or R).

Consider

$$X_1 = \left\{\left(L_{a_1} \mid 0 \quad 0 \quad 0\right)\middle| L_{a_i} \in L_R\right\} \subseteq X,$$

$$X_2 = \left\{\left(0 \mid L_{a_2} \quad 0 \quad 0\right)\middle| L_{a_i} \in L_R\right\} \subseteq X,$$

$$X_3 = \left\{\left(0 \mid 0 \quad L_{a_3} \quad 0\right)\middle| L_{a_i} \in L_R\right\} \subseteq X$$

and

$$X_4 = \left\{\left(0 \mid 0 \quad 0 \quad L_{a_4}\right)\middle| L_{a_i} \in L_R\right\} \subseteq X$$

be four super row vector subspaces of refined labels of X over L_R.

We see $X = \bigcup_{i=1}^{4} X_i$ and $X_i \cap X_j = (0 \mid 0\ 0\ 0)$ if $i \neq j$; $1 \leq i, j \leq 4$.

However X has more than four super row vector subspaces of refined labels.

For take $Y_1 = \left\{\left(L_{a_1} \mid L_{a_2} \quad 0 \quad 0\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 2\right\} \subseteq X$ is a super row vector subspace of refined labels.

Consider $Y_2 = \left\{\left(L_{a_1} \mid 0 \quad 0 \quad L_{a_2}\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 2\right\} \subseteq X$ is a super row vector subspace of X of refined labels.

Take $Y_3 = \left\{\left(0 \mid L_{a_1} \quad L_{a_2} \quad 0\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 2\right\} \subseteq X$ is again a super row vector subspace of X of refined labels over L_R.

DEFINITION 4.3: *Let V be a super row vector space of refined labels over the field L_R (or R). Suppose $W_1, W_2, \ldots, W_m$ are super row vector subspaces of refined labels of V over the field L_R (or R). If $V = \bigcup_{i=1}^{m} W_i$ and $W_i \cap W_j = (0)$ if $i \neq j$, $1 \leq i, j \leq m$ then we say V is a direct union or sum of super row vector subspaces of V.*

If $W_1, \ldots, W_m$ are super row vector subspaces of refined labels of V over L_R (or R) and $V = \bigcup_{i=1}^{m} W_i$ but $W_i \cap W_j \neq (0)$ if $i \neq j$, $1 \leq i, j \leq m$ then we say V is a pseudo direct sum of super vector subspaces of refined labels of V over L_R (or R).

Example 4.8: Let

$$V = \left\{\left(L_{a_1} \quad L_{a_2} \mid L_{a_3} \mid L_{a_4} \quad L_{a_5} \quad L_{a_6} \mid L_{a_7}\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 7\right\}$$

be a super row vector space of refined labels over L_R (or R). Consider

$$W_1 = \left\{\left(L_{a_1} \quad L_{a_2} \mid 0 \mid 0 \quad 0 \quad 0 \mid 0\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 2\right\} \subseteq V,$$

$$W_2 = \left\{\left(L_{a_1} \quad 0 \mid L_{a_2} \mid L_{a_3} \quad 0 \quad 0 \mid L_{a_4}\right) \middle| L_{a_i} \in L_R; 1 \le i \le 4\right\} \subseteq V,$$

$$W_3 = \left\{\left(L_{a_1} \quad 0 \mid 0 \mid L_{a_3} \quad L_{a_4} \quad 0 \mid 0\right) \middle| L_{a_i} \in L_R; 1 \le i \le 3\right\} \subseteq V,$$

$$W_4 = \left\{\left(L_{a_1} \quad 0 \mid L_{a_2} \mid 0 \quad L_{a_4} \quad L_{a_3} \mid L_{a_5}\right) \middle| L_{a_i} \in L_R; 1 \le i \le 5\right\} \subseteq$$

V and

$$W_5 = \left\{\left(0 \quad L_{a_1} \mid L_{a_2} \mid 0 \quad 0 \quad 0 \mid L_{a_5}\right) \middle| L_{a_i} \in L_R; 1 \le i \le 3\right\} \subseteq V$$

be super row vector subspaces of refined labels of V over L_R.

Clearly V = $\bigcup_{i=1}^{5} W_i$ but $W_i \cap W_j \neq (0\ 0 \mid 0 \mid 0\ 0\ 0 \mid 0)$ if $i \neq j$, $1 \le i, j \le 5$.

Thus V is only a pseudo direct sum of W_i, i = 1, 2, …, 5 and is not a direct sum. Now we see the supervector space of refined labels as in case of other vector spaces can be written both as a direct sum as well as pseudo direct sum of super vector subspaces of refined labels over L_R (or R).

Example 4.9: Let V =

$$\left\{\left(L_{a_1} \quad L_{a_2} \mid L_{a_3} \quad L_{a_5} \mid L_{a_6} \quad L_{a_7} \quad L_{a_8} \quad L_{a_4}\right) \middle| L_{a_i} \in L_R; 1 \le i \le 8\right\}$$

be a super vector.

Consider

$$W_1 = \left\{\left(L_{a_1} \quad 0 \mid 0 \quad 0 \mid 0 \quad 0 \quad 0 \quad 0\right) \middle| L_{a_i} \in L_R\right\} \subseteq V,$$

$$W_2 = \left\{\left(0 \quad L_{a_1} \mid 0 \quad 0 \mid 0 \quad 0 \quad 0 \quad 0\right) \middle| L_{a_i} \in L_R\right\} \subseteq V,$$

$$W_3 = \left\{\left(0 \quad 0 \mid L_{a_1} \quad 0 \mid 0 \quad 0 \quad 0 \quad 0\right) \middle| L_{a_i} \in L_R\right\} \subseteq V,$$

$$W_4 = \left\{\left(0 \quad 0 \mid 0 \quad L_{a_1} \mid 0 \quad 0 \quad 0 \quad 0\right) \middle| L_{a_i} \in L_R\right\} \subseteq V,$$

$$W_5 = \left\{\left(0 \quad 0 \mid 0 \quad 0 \mid L_{a_1} \quad 0 \quad 0 \quad 0\right) \middle| L_{a_i} \in L_R\right\} \subseteq V,$$

$$W_6 = \left\{\left(0 \quad 0 \mid 0 \quad 0 \mid 0 \quad L_{a_1} \quad 0 \quad 0\right) \middle| L_{a_i} \in L_R\right\} \subseteq V,$$

$$W_7 = \left\{\left(0 \quad 0 \mid 0 \quad 0 \mid 0 \quad 0 \quad L_{a_1} \quad 0\right) \middle| L_{a_i} \in L_R\right\} \subseteq V \text{ and}$$

$$W_8 = \left\{\left(0 \quad 0 \mid 0 \quad 0 \mid 0 \quad 0 \quad 0 \quad L_{a_1}\right) \middle| L_{a_i} \in L_R\right\} \subseteq V;$$

are super row vector subspaces of V over the field L_R. Clearly $V = \bigcup_{i=1}^{8} W_i$ but $W_i \cap W_j = (0\ 0 \mid 0\ 0 \mid 0\ 0\ 0\ 0)$ if $i \neq j$, $1 \leq i, j \leq 8$.

V is a direct sum of super row vector subspaces $W_1, W_2, \ldots, W_8$ of V over L_R.

Consider

$$W_1 = \left\{\left(L_{a_1} \ \ L_{a_2} \mid 0 \ \ 0 \mid 0 \ \ 0 \ \ 0 \ \ L_{a_3}\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 3\right\} \subseteq V,$$

$$W_2 = \left\{\left(L_{a_1} \ \ 0 \mid L_{a_2} \ \ L_{a_3} \mid 0 \ \ 0 \ \ 0 \ \ 0\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 3\right\} \subseteq V,$$

$$W_3 = \left\{\left(0 \ \ L_{a_1} \mid 0 \ \ 0 \mid L_{a_2} \ \ L_{a_3} \ \ 0 \ \ 0\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 3\right\} \subseteq V$$

and

$$W_4 = \left\{\left(L_{a_1} \ \ 0 \mid L_{a_2} \ \ 0 \mid L_{a_3} \ \ 0 \ \ L_{a_4} \ \ 0\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 3\right\} \subseteq V$$

be super row vector subspaces of V over the refined field L_R (or R). Clearly $V = \bigcup_{i=1}^{4} W_i$ is a pseudo direct sum of super row vector subspaces of V over the field L_R as $W_i \cap W_j \neq (0\ 0 \mid 0\ 0 \mid 0\ 0\ 0\ 0)$ if $i \neq j$, $1 \leq i, j \leq 4$.

Now having seen examples of direct sum and pseudo direct sum of super row vector subspaces of V over L_R we now proceed onto define linear operators, basis and linear transformation of super row vector spaces of refined labels over L_R.

It is pertinent to mention here that $L_R \cong R$ so whether we work on R or L_R it is the same.

DEFINITION 4.4 : *Let V =*
$\left\{\left(L_{a_1} \mid L_{a_2} \quad \ldots \mid L_{a_{n-1}} \quad L_{a_n}\right) \middle| L_{a_i} \in L_R; 1 \le i \le n\right\}$ *be a super row vector space of refined labels over the field L_R. We say a subset of elements $\{x_1, x_2, \ldots, x_t\}$ in V are linearly dependent if there exists scalar labels $L_{a_1} \quad \ldots \quad L_{a_t}$ in L_R not all of them equal to zero such that $L_{a_1} x_1 + L_{a_2} x_2 + \ldots + L_{a_t} x^t = 0$. A set which is not linearly dependent is called linearly independent.*

We say for the super row vector space of refined labels V =
$\left\{\left(L_{a_1} \quad L_{a_2} \mid L_{a_3} \quad L_{a_4} \quad L_{a_5} \mid \ldots \mid L_{a_{n-1}} \quad L_{a_n}\right) \middle| L_{a_i} \in L_R; 1 \le i \le n\right\}$
a subset B of V to be a basis of V if

(1) B is a linearly independent subset of V.

(2) B generates or spans V over L_R.

We will give examples of them. It is pertinent to mention in L_R; L_{m+1} is the unit element of the field L_R.

Example 4.10: Let V =
$\left\{\left(L_{a_1} \quad L_{a_2} \mid L_{a_3} \mid L_{a_4} \quad L_{a_5} \quad L_{a_6} \mid L_{a_7}\right) \middle| L_{a_i} \in L_R; 1 \le i \le 7\right\}$ be a super row vector space of refined labels over the field L_R.

Consider the set

B = {(L_{m+1}, 0 | 0 | 0 0 0 | 0), (0, L_{m+1} | 0 | 0 0 0 | 0), (0 0 | L_{m+1} | 0 0 0 | 0), (0 0 | 0 | L_{m+1} 0 0 | 0), (0 0 | 0 | 0 L_{m+1} 0 | 0), (0 0 | 0 | 0 0 L_{m+1} | 0), (0 0 | 0 | 0 0 0 |L_{m+1})} ⊆ V is a super basis of over L_R. Clearly V is finite dimensional over L_R.

Consider P = $\left(L_{a_1} \quad L_{a_2} \mid 0 \mid 0 \quad 0 \quad 0 \mid L_{a_7}\right)$,
$\left(0 \quad 0 \mid L_{a_2} \mid 0 \quad 0 \quad 0 \mid 0\right)$, $\left(0 \quad 0 \mid 0 \mid L_{a_3} \quad 0 \quad 0 \mid 0\right)$,
$\left(0 \quad 0 \mid 0 \mid 0 \quad L_{a_4} \quad L_{a_5} \mid 0\right)$ ⊆ V; P is only a super linearly independent subset of V but is not a basis of V.

Example 4.11: Let V = $\left\{\left(L_{a_1} \mid L_{a_2} \quad L_{a_3}\right) \middle| L_{a_i} \in L_R ; 1 \le i \le 3;\right\}$ be a super row vector space of refined labels over L_R. Consider B = {(L_{m+1} | 0 0), (0, | L_{m+1} 0), (0 | 0 L_{m+1})} $\subseteq$ V is a super basis of V over L_R.

Clearly dimension of V is three over L_R. Consider M $=\left\{\left(L_{a_1} \mid 0 \quad 0\right), \left(0 \mid L_{a_1} \quad L_{a_2}\right)\right\} \subseteq$ V; M is only a linearly independent set of V over L_R and is not a basis of V over L_R.

Take K = $\left\{\left(L_{a_1} \mid 0 \quad L_{a_2}\right), \left(L_{a_1} \mid L_{a_2} \quad 0\right), \left(0 \mid L_{a_1} \quad L_{a_2}\right),\right.$ $\left.\left(L_{a_1} \mid 0 \quad 0\right), \left(0 \mid L_{a_2} \quad 0\right)\right\} \subseteq$ V, K is a linearly dependent subset of V. Consider B = $\left\{\left(0 \mid L_{a_1} \quad 0\right), \left(0 \mid L_{a_1} \quad L_{a_2}\right), \left(0 \mid 0 \quad L_{a_2}\right)\right\} \subseteq$ V. B is also a linearly dependent subset of V over L_R.

Now having seen examples of linearly dependent subset, linearly independent subset and not a basis and linear independent subset of V which is a super basis of V, we now proceed onto define the linearly transformation of super row vector spaces of refined labels over L_R.

Let V and W be two super row vector space of refined labels over L_R. Let T be a map from V to W we say T : V $\to$ W is a linear transformation if T (cv + u) = cT (v) + T (u) for all u, v $\in$ V and c $\in$ L_R.

Example 4.12: Let V = $\left\{\left(L_{a_1} \mid L_{a_2} \quad L_{a_3} \mid L_{a_4} \quad L_{a_5} \quad L_{a_6}\right) \middle| L_{a_i} \in L_R ; 1 \le i \le 6\right\}$ be a super row vector space of refined labels over L_R. W = $\left\{\left(L_{a_1} \quad L_{a_2} \mid L_{a_3} \mid L_{a_4} \quad L_{a_5} \quad L_{a_6} \quad L_{a_7} \quad L_{a_8}\right) \middle| L_{a_i} \in L_R ; 1 \le i \le 8\right\}$ be a super row vector space of refined labels over L_R. Define a map T : V $\to$ W by T $\left(L_{a_1} \mid L_{a_2} \quad L_{a_3} \mid L_{a_4} \quad L_{a_5} \quad L_{a_6}\right)$ = $\left(L_{a_1} \quad 0 \mid L_{a_2} \quad 0 \mid L_{a_3} \quad L_{a_4} \quad L_{a_5} \quad L_{a_6} \quad 0\right)$; T is easily verified to be a linear transformation.

Let $P : V \to W$ be a map such that

$\left(L_{a_1} \mid L_{a_2} \quad L_{a_3} \mid L_{a_4} \quad L_{a_5} \quad L_{a_6}\right) =$

$\left(L_{a_1} \quad L_{a_2} \mid L_{a_3} \mid L_{a_4} \quad L_{a_5} \quad L_{a_6} \quad 0 \quad 0\right)$; It is easily verified P is a linear transformation of super row vector space V into W.

Example 4.13: Let V =

$\left\{\left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \mid L_{a_4} \quad L_{a_5} \mid L_{a_6} \mid L_{a_7} \quad L_{a_8} \quad L_{a_9} \quad L_{a_{10}}\right) \middle| \right.$

$\left. L_{a_i} \in L_R; 1 \leq i \leq 10\right\}$ be a super row vector space of refined labels over the field L_R. Let T be a map from V to V defined by T $\left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \mid L_{a_4} \quad L_{a_5} \mid L_{a_6} \mid L_{a_7} \quad L_{a_8} \quad L_{a_9} \quad L_{a_{10}}\right) =$ $\left(L_{a_1} \quad 0 \quad L_{a_3} \mid 0 \quad L_{a_5} \mid 0 \mid L_{a_7} \quad 0 \quad L_{a_9} \quad 0\right)$; T is a linear operator on V. It is clear T is not one to one. Ker T is a non zero subspace of V. Consider a map $P : V \to V$ defined by P $\left(\left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \mid L_{a_4} \quad L_{a_5} \mid L_{a_6} \mid L_{a_7} \quad L_{a_8} \quad L_{a_9} \quad L_{a_{10}}\right)\right) =$ $\left(L_{a_3} \quad L_{a_2} \quad L_{a_1} \mid L_{a_5} \quad L_{a_4} \mid L_{a_6} \mid L_{a_{10}} \quad L_{a_9} \quad L_{a_8} \quad L_{a_7}\right)$. P is a linear operator with trivial kernel.

Now suppose V =

$\left\{\left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \mid L_{a_4} \mid L_{a_5} \quad L_{a_6} \quad L_{a_7} \quad L_{a_8}\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 8\right\}$

be a super row vector space of refined labels over the field L_R.

Consider W =

$\left\{\left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \mid 0 \mid 0 \quad 0 \quad 0 \quad L_{a_8}\right) \middle| L_{a_i} \in L_R; i = 1,2,3,8\right\} \subseteq V$;

W is a super row vector subspace of V of refined labels over the field L_R.

Let $T : V \to V$ if $T(W) \subseteq W$, W is invariant under T, we can illustrate this situation by some examples.

$$T\left(\left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \mid L_{a_4} \mid L_{a_5} \quad L_{a_6} \quad L_{a_7} \quad L_{a_8}\right)\right) =$$

$\left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \mid 0 \mid 0 \quad 0 \quad 0 \quad L_{a_8}\right)$;is such that T(W)$\subseteq$ W.

Suppose M =

$\left\{\left(L_{a_1} \quad 0 \quad L_{a_2} \mid 0 \mid L_{a_3} \quad 0 \quad L_{a_4} \quad 0\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 4\right\} \subseteq V;$

then M is a super row vector subspace of V of refined labels over L_R. Define $\eta : V \to V$ by

$$\eta\left(\left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \mid L_{a_4} \mid L_{a_5} \quad L_{a_6} \quad L_{a_7} \quad L_{a_8}\right)\right) =$$

$\left(L_{a_1} \quad 0 \quad L_{a_2} \mid 0 \mid L_{a_3} \quad 0 \quad L_{a_4} \quad 0\right)$; η is a linear transformation and M is invariant under η as η (M) $\subseteq$ M.

As in case of usual vector spaces in case of super row vector space of refined labels over the field L_R we can have the following theorem.

THEOREM 4.1: *Let V and W be two finite dimensional super vector spaces of refined labels over the field L_R such that dim V = dim W. If T is a linear transformation from V into W, the following are equivalent.*

- *(i) T is invertible*
- *(ii) T is non singular*
- *(iii) T is onto, that is range of T is W.*

The proof is left as an exercise to the reader.

Further we can as in case of usual vector space define in case of super row vector space of refined labels also the following results.

THEOREM 4.2: *If $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis for V then $\{T\alpha_1, T\alpha_2, \ldots, T\alpha_n\}$ is a basis for W. (dim V = dim W = n is assumed), V a finite dimensional super vector space over L_R and T_a is a linear transformation from V to W.*

This proof is also direct and hence left as an exercise to the reader.

THEOREM 4.3: *Let V and W be two super row vector spaces of same dimension over the real field L_R. There is some basis $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ for V such that $\{T\alpha_1, T\alpha_2, \ldots, T\alpha_n\}$ is a basis for W.*

The proof of this results is direct.

THEOREM 4.4: *Let T be a linear transformation from V into W, V and W super row vector spaces of refined labels over the field L_R then T is non singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of W.*

We make use of the following theorem to prove the theorem 4.4.

THEOREM 4.5: *Let V and W be any two super row vector spaces of refined labels over the field L_R and let T be a linear transformation from V into W. If T is invertible then the inverse function T^{-1} is a linear transformation from W onto V.*

Also as in case of usual vector space we see in case of super row vector space V of refined labels over a field F = L_R if T_1, T_2 and P are linear operators on V and for $c \in L_R$ we have

(i) $IP = PI = P$
(ii) $P(T_1 + T_2) = PT_1 + PT_2$
(iii) $(T_1 + T_2) P = T_1P + T_2P$
(iv) $c(PT_1) = (cP)T_1 = P(cT_1)$.

All the results can be proved without any difficulty and hence is left as an exercise to the reader.

Also if V, W and Z are super row vector spaces of refined labels over the field L_R and $T : V \to W$ be a linear transformation of V into W and P a linear transformation of W into Z, then the composed function PT defined by $(PT)(\alpha) = P(T(\alpha))$ is a linear transformation from V into Z. This proof is also direct and simple and hence is left as an exercise to the reader.

Now having studied properties about super row vector spaces of refined labels we now proceed onto define super column vector spaces of refined labels over L_R.

DEFINITION 4.5: *Let* $V = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline \vdots \\ \hline L_{a_{n-1}} \\ L_{a_n} \end{array}\right] \middle| L_{a_i} \in L_R; 1 \leq i \leq n \right\}$ *be a super column vector space of refined labels over the field L_R (or R); clearly V is an abelian group under addition.*

We will illustrate this situation by some examples.

Example 4.14: Let $V = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ L_{a_8} \end{array}\right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 8 \right\}$ be group under addition. V is a super column vector space of refined labels over the field L_R (or R).

Example 4.15: Let $V = \left\{ \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ L_{a_4} \\ L_{a_5} \end{array}\right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 5 \right\}$ be super column vector space of refined labels over the field L_R.

Example 4.16: Let V = $\left\{\left.\begin{bmatrix} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \end{bmatrix}\right| L_{a_i} \in L_R; 1 \le i \le 7\right\}$ be a super column vector space of refined labels over the field L_R (or R).

Example 4.17: Let V = $\left\{\left.\begin{bmatrix} L_{a_1} \\ \hline L_{a_2} \\ \hline L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ \hline L_{a_6} \\ \hline L_{a_7} \end{bmatrix}\right| L_{a_i} \in L_R; 1 \le i \le 7\right\}$ be a super column vector space over the field L_R.

We can define the super column vector subspace of V, V a super column vector space over L_R.

We will only illustrate the situation by some examples.

Example 4.18: Let V = $\left\{\left.\begin{bmatrix} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ L_{a_4} \\ L_{a_5} \\ \hline L_{a_6} \\ L_{a_7} \end{bmatrix}\right| L_{a_i} \in L_R; 1 \leq i \leq 7\right\}$ be a super column vector space over L_R.

Let $W_1 = \left\{\left.\begin{bmatrix} L_{a_1} \\ 0 \\ \hline L_{a_2} \\ 0 \\ L_{a_3} \\ \hline 0 \\ L_{a_4} \end{bmatrix}\right| L_{a_i} \in L_R; 1 \leq i \leq 4\right\} \subseteq V$ be a super column vector subspace of V over L_R.

$W_2 = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ \hline L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ \hline 0 \\ 0 \end{bmatrix}\right| L_{a_i} \in L_R; 1 \leq i \leq 3\right\} \subseteq V$ is a super column vector subspace of V over L_R.

$$W_3 = \left\{ \left[\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline L_{a_1} \\ L_{a_2} \end{array} \right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 2 \right\} \subseteq V \text{ is a super column}$$

vector subspace of V of refined labels over L_R.

Example 4.19: Let $V = \left\{ \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ L_{a_4} \\ \hline L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ \hline L_{a_8} \\ L_{a_9} \end{array} \right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 9 \right\}$ be a super column vector space of refined labels over L_R.

Take $W_1 = \left\{ \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \hline L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ \hline 0 \\ L_{a_4} \end{array} \right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 4 \right\} \subseteq V$; is a super column vector subspace of refined labels over L_R.

$W_2 = \left\{ \left[\begin{array}{c} 0 \\ L_{a_1} \\ L_{a_2} \\ 0 \\ \hline 0 \\ L_{a_3} \\ 0 \\ \hline 0 \\ 0 \end{array} \right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 3 \right\} \subseteq V$ is a super column vector subspace of refined labels over L_R.

Example 4.20: Let V = $\left\{ \begin{bmatrix} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ L_{a_4} \\ L_{a_5} \\ \hline L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \\ L_{a_9} \\ \hline L_{a_{10}} \end{bmatrix} \middle| L_{a_i} \in L_R; 1 \leq i \leq 10 \right\}$ be a super column vector space of refined labels over L_R.

Consider W_1 = $\left\{ \begin{bmatrix} L_{a_1} \\ 0 \\ L_{a_2} \\ 0 \\ L_{a_3} \\ \hline 0 \\ \hline L_{a_4} \\ 0 \\ 0 \\ \hline L_{a_5} \end{bmatrix} \middle| L_{a_i} \in L_R; 1 \leq i \leq 5 \right\} \subseteq V$, is a super column vector subspace of refined labels over L_R.

Take $W_2 = \left\{ \left[\begin{array}{c} 0 \\ L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ 0 \\ \hline L_{a_4} \\ \hline L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ \hline 0 \end{array} \right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 7 \right\} \subseteq V$ is a super column vector subspace of refined labels over L_R.

Example 4.21: Let $V = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \\ L_{a_9} \\ L_{a_{10}} \\ \hline L_{a_{11}} \\ L_{a_{12}} \end{array} \right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 12 \right\}$ be a super column vector space of refined labels over the field L_R. Consider the following super column vector subspaces of refined labels over L_R of V.

$$W_1 = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline 0 \\ 0 \\ 0 \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ \hline 0 \\ 0 \end{array}\right] \middle| L_{a_i} \in L_R; 1 \le i \le 7 \right\} \subseteq V \text{ is a super column}$$

vector subspace of V over L_R.

$$W_2 = \left\{ \left[\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ \hline L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ \hline L_{a_4} \\ 0 \end{array}\right] \middle| L_{a_i} \in L_R; 1 \le i \le 4 \right\} \subseteq V \text{ is a super column}$$

vector subspace of V over L_R.

$$W_3 = \left\{ \left[\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ L_{a_1} \end{array}\right] \middle| \; L_{a_1} \in L_R \right\} \subseteq V$$ is a super column vector

subspace of refined labels of V over L_R. We see $V = \bigcup_{i=1}^{3} W_i$ and

$$W_i \cap W_j = \left[\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right] \text{ if } i \neq j,\ 1 \leq i, j \leq 3.$$

Example 4.22: Let V = $\left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \end{array} \right] \middle| L_{a_i} \in L_R; 1 \le i \le 6 \right\}$ be a super column vector space of refined labels on L_R. Take W_1 = $\left\{ \left[\begin{array}{c} 0 \\ \hline L_{a_1} \\ L_{a_2} \\ \hline 0 \\ 0 \\ 0 \end{array} \right] \middle| L_{a_i} \in L_R; 1 \le i \le 2 \right\} \subseteq$ V, W_2 = $\left\{ \left[\begin{array}{c} L_{a_1} \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array} \right] \middle| L_{a_i} \in L_R \right\} \subseteq$ V

and W_3 = $\left\{ \left[\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ \hline L_{a_1} \\ L_{a_2} \\ L_{a_3} \end{array} \right] \middle| L_{a_i} \in L_R; 1 \le i \le 3 \right\} \subseteq$ V be three super column vector subspace of refined labels of V over L_R.

We see V = $\bigcup_{i=1}^{3} W_i$ and $W_i \cap W_j$ = $\left[\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array} \right]$ if i ≠ j, 1 ≤ i, j ≤ 3.

Thus V is a direct union of super vector column subspaces of V over L_R.

Example 4.23: Let $V = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \end{array}\right] \middle| L_{a_i} \in L_R; 1 \le i \le 8 \right\}$ be a super column vector space of refined labels over L_R (or R). Consider

$$W_1 = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline 0 \\ 0 \\ 0 \\ \hline L_{a_4} \\ 0 \end{array}\right] \middle| L_{a_i} \in L_R; 1 \le i \le 4 \right\} \subseteq V,$$

be a super column vector subspace of refined labels of V over L_R,

$$W_2 = \left\{ \left[\begin{array}{c} 0 \\ \hline L_{a_1} \\ 0 \\ \hline L_{a_2} \\ L_{a_3} \\ L_{a_4} \\ \hline 0 \\ 0 \end{array}\right] \middle| L_{a_i} \in L_R; 1 \le i \le 4 \right\} \subseteq V$$

be a super column vector subspace of refined labels of V over L_R.

$$W_3 = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline 0 \\ L_{a_2} \\ \hline 0 \\ L_{a_3} \\ 0 \\ \hline 0 \\ L_{a_4} \end{array}\right] \middle| \, L_{a_i} \in L_R; 1 \leq i \leq 4 \right\} \subseteq V \text{ be a super column}$$

vector subspace of refined labels of V over L_R.

$$W_4 = \left\{ \left[\begin{array}{c} 0 \\ \hline L_{a_1} \\ L_{a_2} \\ \hline 0 \\ L_{a_3} \\ L_{a_4} \\ \hline L_{a_5} \\ L_{a_6} \end{array}\right] \middle| \, L_{a_i} \in L_R; 1 \leq i \leq 6 \right\} \subseteq V \text{ be a super column}$$

vector subspace of refined labels over L_R of V.

Clearly $V = \bigcup_{i=1}^{4} W_i$ and $W_i \cap W_j \neq \left[\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right]$; $i \neq j$, $1 \leq i \leq 4$.

Thus V is a pseudo direct sum of super column vector subspaces of V over L_R.

It is pertinent to mention that we do not have super column linear algebras of V of refined labels over L_R. However we can also have subfield super column vector subspace of refined labels over L_R.

We will illustrate this by some examples. Just we know L_Q is a field of refined labels and $L_Q \subseteq L_R$. Thus we can have subfield super column vector subspaces as well as subfield super row vector subspaces of refined labels over the subfield L_Q of L_R (or Q of R).

Example 4.24: Let V = $\left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ \hline L_{a_8} \end{array} \right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 8 \right\}$ be a super column vector space of refined labels over the field L_R (or R).

Take M = $\left\{ \left[\begin{array}{c} L_{a_1} \\ \hline 0 \\ 0 \\ \hline L_{a_2} \\ \hline 0 \\ 0 \\ 0 \\ \hline L_{a_3} \end{array} \right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 3 \right\} \subseteq$ V be a super column vector space of refined labels over the refined labels field L_Q. ($L_Q \subseteq L_R$). M is thus the subfield super column vector subspace of refined labels over the subfield L_Q of L_R.

Example 4.25: Let M =

$$\left\{ \left(L_{a_1} \quad L_{a_2} \mid L_{a_3} \mid L_{a_4} \quad L_{a_5} \quad L_{a_6} \mid L_{a_7} \quad L_{a_8} \quad L_{a_9} \quad L_{a_{10}} \right) \middle| \right.$$

$\left. L_{a_i} \in L_R; 1 \leq i \leq 10 \right\}$ be a super row vector space of refined labels over L_R.

Take P =

$$\left\{ \left(L_{a_1} \quad L_{a_2} \mid L_{a_3} \mid L_{a_4} \quad L_{a_5} \quad L_{a_6} \mid L_{a_7} \quad L_{a_8} \quad L_{a_9} \quad L_{a_{10}} \right) \middle| \right.$$

$\left. L_{a_i} \in L_Q; 1 \leq i \leq 10 \right\} \subseteq$ M; P is a super row vector space of refined labels over L_Q; thus P is a subfield super row vector subspace of V of refined labels over the subfield L_Q of L_R. We have several such subfield vector subspaces of V.

Example 4.26: Let V = $\left\{ \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ L_{a_4} \\ L_{a_5} \\ \hline L_{a_6} \\ L_{a_7} \\ \hline L_{a_8} \end{array} \right] \middle| L_{a_i} \in L_Q; 1 \leq i \leq 8 \right\}$ be a super column vector space of refined labels over the field L_Q. Clearly V has no subfield super column vector subspaces of refined labels of V over the subfield over L_Q as L_Q has no subfields. In this case we call such spaces as pseudo simple super column vector subspaces of refined labels of V over L_Q. However V has several super column vector subspaces of refined labels of

V over L_Q. For instance take K = $\left\{ \left[\begin{array}{c} L_{a_1} \\ 0 \\ \hline L_{a_2} \\ 0 \\ L_{a_3} \\ \hline 0 \\ L_{a_4} \\ \hline 0 \end{array}\right] \middle| L_{a_i} \in L_R; 1 \le i \le 4 \right\} \subseteq$

V, K is a super column vector subspace of V over the field L_R of refined labels.

Example 4.27: Let V =

$\left\{ \left(L_{a_1} \quad L_{a_2} \mid L_{a_3} \quad L_{a_4} \quad L_{a_5} \mid L_{a_6} \quad L_{a_7} \quad L_{a_8} \quad L_{a_9} \mid L_{a_{10}} \right) \right|$

$L_{a_i} \in L_R; 1 \le i \le 10 \}$ be a super row vector space of refined labels over the field L_Q.

Clearly L_Q has no proper subfields other than itself as $L_Q \cong Q$. Hence V cannot have subspaces which are subfield super row vector subspace of refined labels over a subfield of L_Q so V is a pseudo simple super row vector space of refined labels over L_Q.

Example 4.28: Let V = $\left\{\left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ \hline L_{a_8} \\ L_{a_9} \\ \hline L_{a_{10}} \end{array}\right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 10\right\}$ be a super column vector space of refined labels over L_Q. V is a pseudo simple super column vector space of refined labels. However V has several super column vector subspaces of refined labels over L_Q of V.

For instance T = $\left\{\left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline L_{a_4} \end{array}\right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 4\right\} \subseteq$ V is a super column vector subspace of V of refined labels over L_Q.

Now we can as in case of other vector spaces define basis linearly independent element linearly dependent element linear

operator and linear transformation. This task is left as an exercise to the reader. However all these situations will be described by appropriate examples.

Example 4.29: Let V = $\left\{\left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \end{array}\right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 6\right\}$ be a super column vector space of refined labels over the refined field L_R.

Consider the set B =

$$\left\{\left[\begin{array}{c} L_{m+1} \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ \hline L_{m+1} \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ \hline 0 \\ L_{m+1} \\ \hline 0 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ \hline L_{m+1} \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ L_{m+1} \\ 0 \end{array}\right]\right\} \subseteq V.$$

It is easily verified B is a linearly independent subset of V but is not a basis of V. We see B cannot generate V.

Now consider C = $\left\{\left[\begin{array}{c} L_{m+1} \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} L_{a_1} \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ \hline L_{a_1} \\ 0 \\ \hline 0 \\ 0 \\ L_{a_2} \end{array}\right], \left[\begin{array}{c} 0 \\ \hline L_{m+1} \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ \hline L_{a_1} \\ 0 \\ 0 \end{array}\right]\right\} \subseteq$ V; It is easily verifield the subset C of V cannot generate V so is not a basis further C is not a linearly independent subset of V as

we see $\left[\begin{array}{c} L_{m+1} \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right]$ and $\left[\begin{array}{c} L_{a_1} \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right]$ are linearly dependent on each other.

These are more than one pair of elements which are linearly dependent.

Now consider H =

$$\left\{\left[\begin{array}{c} L_{m+1} \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ \hline L_{m+1} \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ \hline 0 \\ L_{m+1} \\ \hline 0 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ \hline L_{m+1} \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ L_{m+1} \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ L_{m+1} \end{array}\right]\right\} \subseteq V \text{ is a basis}$$

of V and the space is of finite dimension and dimension of V over L_R is six.

Now if we shift the field L_R from L_Q we see the dimension of V over L_Q is infinite.

Thus as in case of usual vector spaces of dimension of a super column vector space of refined labels depends on the field over which they are defined.

This above example shows if V is defined over L_R it is of dimension six but if V is defined over L_Q the dimension of V over L_Q is infinite.

Example 4.30: Let V =

$$\left\{\left(L_{a_1} \quad L_{a_2} \mid L_{a_3} \quad L_{a_4} \quad L_{a_5} \mid L_{a_6} \mid L_{a_7} \quad L_{a_8} \quad L_{a_9} \quad L_{a_{10}}\right) \middle| \; L_{a_i} \in L_R; 1 \le i \le 10\right\}$$

be a super row vector space of refined labels over the refined field L_Q. Clearly V is of infinite

dimension over L_Q. Further V has a linearly independent as well as linearly dependent subset over L_Q.

Consider B = $\{(L_{a_1} \quad 0 \mid 0 \quad 0 \quad 0 \mid 0 \mid 0 \quad 0 \quad 0 \quad 0),$
$(0 \quad 0 \mid L_{a_2} \quad 0 \quad 0 \mid 0 \mid 0 \quad 0 \quad 0 \quad 0),$
$(0 \quad 0 \mid 0 \quad 0 \quad 0 \mid L_{a_3} \mid 0 \quad 0 \quad 0 \quad 0),$
$(0 \quad 0 \mid 0 \quad 0 \quad 0 \mid 0 \mid L_{a_5} \quad 0 \quad L_{a_6} \quad 0)$
$(0 \quad 0 \mid 0 \quad 0 \quad 0 \mid 0 \mid 0 \quad L_{a_7} \quad 0 \quad L_{a_8}) \mid L_{a_i} \in L_R; 1 \leq i \leq 8\}$
$\subseteq$ V, B is a linearly independent subset of V over L_Q.

Take P = $\{(L_{a_1} \quad L_{a_2} \mid 0 \quad 0 \quad 0 \mid 0 \mid 0 \quad 0 \quad 0 \quad 0),$
$(0 \quad 0 \mid L_{a_1} \quad L_{a_2} \quad 0 \mid 0 \mid 0 \quad 0 \quad 0 \quad 0),$
$(0 \quad L_{a_2} \mid 0 \quad 0 \quad L_{a_2} \mid 0 \mid 0 \quad 0 \quad 0 \quad 0),$
$(L_{a_1} \quad 0 \mid L_{a_1} \quad 0 \quad 0 \mid L_{a_4} \mid 0 \quad 0 \quad 0 \quad 0),$
$(0 \quad 0 \mid 0 \quad 0 \quad 0 \mid 0 \mid 0 \quad 0 \quad 0 \quad L_{a_5})$
$(0 \quad 0 \mid L_{a_1} \quad L_{a_2} \quad L_{a_3} \mid 0 \mid 0 \quad 0 \quad 0 \quad 0)\} \subseteq$ V, P is a linearly dependent subset of V over L_Q. We have seen though V is of infinite dimension over L_Q still V can have both linearly dependent subset as well as linearly independent subset.

Example 4.31: Let V =
$\{(L_{a_1} \mid L_{a_2} \mid L_{a_3} \mid L_{a_4} \mid L_{a_5}) \mid L_{a_i} \in L_Q; 1 \leq i \leq 5\}$ be a super row vector space of refined labels over the field L_Q. Clearly V is finite dimensional and dimension of V over L_Q is given by B = $\{(L_{m+1} \mid 0 \mid 0 \mid 0 \mid 0), (0 \mid L_{m+1} \mid 0 \mid 0 \mid 0), (0 \mid 0 \mid L_{m+1} \mid 0 \mid 0),$
$(0 \mid 0 \mid 0 \mid L_{m+1} \mid 0), (0 \mid 0 \mid 0 \mid 0 \mid L_{m+1})\} \subseteq$ V, which is the basis of V over L_Q.

Consider M =

$$\left\{\left(L_{a_1} \mid 0 \mid 0 \mid 0 \mid 0\right), \left(L_{m+1} \mid 0 \mid 0 \mid 0 \mid 0\right), \left(0 \mid L_{a_1} \mid L_{a_2} \mid 0 \mid 0\right),\right.$$

$$\left(0 \mid L_{m+1} \mid L_{m+1} \mid 0 \mid 0\right), \left(0 \mid 0 \mid 0 \mid L_{a_1} \mid 0\right), \left(0 \mid 0 \mid 0 \mid L_{b_1} \mid 0\right) \Big|$$

$\left. L_{b_1}, L_{a_i} \in L_Q; 1 \le i \le 2\right\} \subseteq$ V is a subset of V but is clearly not a linearly independent set only a linearly dependent subset of V.

Now we will give a linearly independent subset of V which is not a basis of V.

Consider T =

$$\left\{\left(L_{a_1} \mid 0 \mid 0 \mid 0 \mid 0\right), \left(0 \mid 0 \mid L_{m+1} \mid 0 \mid 0\right), \left(0 \mid 0 \mid 0 \mid L_{m+1} \mid L_{m+1}\right)\right\}$$

in V; clearly T is not a basis but T is a linearly independent subset of V over L_Q.

Now having seen the concept of basis, linearly independent subset and linearly dependent subset we now proceed onto define the notion of linear transformation and linear operator in case of the refined space of labels over L_R (or L_Q).

DEFINITION 4.6: *Let V and W be two super row vector spaces of refined labels defined over the refined field L_R (or L_Q). Let T be a map from V into W. If $T(c\alpha + \beta) = cT(\alpha) + T(\beta)$ for all c $\in L_R$ (or L_Q) and $\alpha, \beta \in V$ then we define T to be a linear transformation from V into W.*

Note if W = V then we define the linear transformation to be a linear operator on V.

We will illustrate these situations by some examples.

Example 4.32: Let V =

$\left\{\left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \mid L_{a_4} \mid L_{a_5} \quad L_{a_6}\right) \Big| L_{a_i} \in L_R; 1 \le i \le 6\right\}$ and W

$= \left\{\left(L_{a_1} \mid L_{a_2} \quad L_{a_3} \mid L_{a_4} \quad L_{a_5} \quad L_{a_6} \mid L_{a_7} \mid L_{a_8} \quad L_{a_9} \quad L_{a_{10}}\right) \Big|\right.$

$\left. L_{a_i} \in L_R; 1 \le i \le 10\right\}$ be two super row vector spaces of refined labels over L_R.

Define T : V → W be such that

$$T\left(\left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \mid L_{a_4} \mid L_{a_5} \quad L_{a_6}\right)\right)$$

$= \left(L_{a_1} \mid L_{a_2} \quad 0 \mid 0 \quad 0 \quad L_{a_3} \mid L_{a_4} \mid L_{a_5} \quad 0 \quad L_{a_6}\right)$; it is easily verified T is a linear transformation of super row vector space of refined labels over L_R.

It is still interesting to find the notions of invertible linear transformation and find the algebraic structure enjoyed by the collection of all linear transformations from V to W.

Example 4.33: If V = $\left\{ \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ L_{a_6} \end{array}\right] \middle| L_{a_i} \in L_R; 1 \le i \le 6 \right\}$ and

W = $\left\{ \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ \hline L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \\ L_{a_9} \\ \hline L_{a_{10}} \end{array}\right] \middle| L_{a_i} \in L_R; 1 \le i \le 10 \right\}$ be two super column vector spaces of refined labels over L_R.

We can define the map T : V → W by

$$T\left(\begin{bmatrix} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ L_{a_6} \end{bmatrix}\right) = \left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \mid L_{a_4} \mid L_{a_5} \mid L_{a_6} \mid 0 \quad 0 \quad 0 \mid 0\right).$$

It is easily verified T is a linear transformation of V into W. Infact we see kernel T is just the zero super vector subspace of V.

Example 4.34: Let V = $\left\{\left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \\ \hline L_{a_9} \end{array}\right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 9\right\}$ be a super column vector space of refined labels over L_R.

Define $T : V \to V$ by $T = \left(\begin{bmatrix} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \\ \hline L_{a_9} \end{bmatrix}\right) = \begin{bmatrix} L_{a_1} \\ \hline 0 \\ 0 \\ \hline L_{a_2} \\ L_{a_3} \\ L_{a_4} \\ \hline 0 \\ 0 \\ \hline L_{a_5} \end{bmatrix}$;

it is easily verified that T is a linear operator on V and kerT is a non trivial super vector column subspace of V. Clearly T is not one to one.

Interested reader can define one to one maps and study the algebraic structure enjoyed by $\text{Hom}_{L_R}(V,V)$ = {set of all linear operators of V to V}. Further it is informed that the reader can derive all the results and theorems proved in the case of super row vector spaces with appropriate modifications in case of super column vector spaces. This task is a matter of routine and hence is left for the reader as an exercise.

Now we proceed onto define the notion of super vector spaces of $m \times n$ super matrices of refined labels over the field L_R.

DEFINITION 4.7: *Let V = {All $m \times n$ super matrices of refined labels with the same type of partition with entries from L_R}; V is clearly an abelian group under addition. V is a super vector space of $m \times n$ matrices of refined labels over the field L_R (or R or L_Q or Q).*

We will illustrate this situation by some simple examples.

Example 4.35: Let V =

$$\left\{ \left[\begin{array}{c|cc} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array} \right] \middle| \, L_{a_i} \in L_R ; 1 \leq i \leq 12 \right\}$$ be the super matrix

vector space of refined labels over L_R.

Example 4.36: Let M = $$\left\{ \left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} \\ L_{a_{15}} & L_{a_{16}} \\ \hline L_{a_{17}} & L_{a_{18}} \end{array} \right] \middle| \, L_{a_i} \in L_R ; 1 \leq i \leq 18 \right\}$$ be a

super matrix vector space of refined labels over L_R. M is also known as the super column vector vector space of refined labels over L_R. These super matrices in M are also known as the super column vectors (refer chapter I of this book).

Example 4.37: Let T =

$$\left\{ \left[\begin{array}{cc|c|ccc} L_{a_1} & L_{a_4} & L_{a_7} & L_{a_{10}} & L_{a_{13}} & L_{a_{16}} \\ L_{a_2} & L_{a_5} & L_{a_8} & L_{a_{11}} & L_{a_{14}} & L_{a_{17}} \\ L_{a_3} & L_{a_6} & L_{a_9} & L_{a_{12}} & L_{a_{15}} & L_{a_{18}} \end{array} \right] \middle| \, L_{a_i} \in L_R ; 1 \leq i \leq 18 \right\}$$ be a

super matrix of vector space of refined labels over the refined field L_R. We can also call T to be a super row vector – vector space of refined labels over L_R.

Example 4.38: Let K =

$$\left\{ \left(\begin{array}{c|ccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array} \right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 16 \right\}$$ be the super matrix of vector space of refined labels over L_R.

Now it is pertinent to mention here that we cannot construct super linear algebra of refined labels even if the natural order of the super matrix is a square matrix. Thus we find it difficult to obtain linear algebras expect those using L_R or L_Q over L_R or L_Q respectively.

Thus we see if V = L_R = $\{L_{a_i} \mid L_{a_i} \in L_R\}$ then V is a vector space as well as linear algebra over L_R or L_Q however it is not a super linear algebra of refined labels over L_R or L_Q.

Likewise L_Q = $\{L_{a_i} \mid L_{a_i} \in L_Q\}$ is only a linear algebra over L_Q, however it is not super linear algebra over L_Q.

Now we can define the notion of super vector subspaces, basis, linear operator and linear transformation as in case of super row vector spaces or super column vector spaces. This task is left as an exercise to the reader. We only give examples of all these concepts so that the reader has no difficulty in following them.

Example 4.39: Let V = $\left\{\left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} \\ \hline L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} \\ \hline L_{a_{15}} & L_{a_{16}} \end{array}\right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 16\right\}$ be a super matrix of vector space of refined labels over L_R. (super column vector, vector space of refined labels).

Consider M = $\left\{\left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ 0 & 0 \\ \hline L_{a_3} & L_{a_4} \\ 0 & 0 \\ L_{a_5} & L_{a_6} \\ \hline 0 & 0 \\ L_{a_7} & L_{a_8} \\ \hline 0 & 0 \end{array}\right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 8\right\} \subseteq V$, is the super column vector subspace of V over L_R.

Take $P = \left\{ \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \\ \hline 0 & 0 \\ 0 & 0 \\ \hline 0 & 0 \end{array} \right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 6 \right\} \subseteq V$ is a super column vector, vector subspace of V over L_R of V.

Example 4.40: Let K =

$$\left\{ \left(\begin{array}{cc|c|ccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ L_{a_{19}} & L_{a_{20}} & L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \end{array} \right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 24 \right\}$$ be

a super row vector, vector space of refined labels over L_R.

Consider M =

$$\left\{ \left(\begin{array}{cc|c|ccc} 0 & 0 & L_{a_1} & 0 & L_{a_5} & L_{a_9} \\ 0 & 0 & L_{a_2} & 0 & L_{a_6} & L_{a_{10}} \\ 0 & 0 & L_{a_3} & 0 & L_{a_7} & L_{a_{11}} \\ 0 & 0 & L_{a_4} & 0 & L_{a_8} & L_{a_{12}} \end{array} \right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 12 \right\} \subseteq K,$$ is a

super row vector, vector subspace of K over L_R.

We can have many such super row vector, vector subspaces of K over R.

Example 4.41: Let V = $\left\{\left(\begin{array}{c|c} L_{a_1} & L_{a_4} \\ \hline L_{a_2} & L_{a_5} \\ L_{a_3} & L_{a_6} \end{array}\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 6\right\}$ be a super matrix vector space of refined labels over the field L_R (or R).

Example 4.42: Let P =

$$\left\{\left(\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \end{array}\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 18\right\}$$

be a super matrix vector space of refined labels over the field L_R. Take M =

$$\left\{\left(\begin{array}{cc|c} 0 & 0 & L_{a_1} \\ 0 & 0 & L_{a_2} \\ \hline L_{a_3} & L_{a_4} & 0 \\ \hline 0 & 0 & L_{a_5} \\ L_{a_6} & L_{a_7} & 0 \\ 0 & 0 & L_{a_8} \end{array}\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 8\right\}$$

$\subseteq$ P is a super matrix vector subspace of P over L_R.

Example 4.43: Let V = $\left\{\left(\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \end{array}\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 9\right\}$

be a super matrix vector space of refined labels over L_R.

Take $M = \left\{ \left(\begin{array}{cc|c} 0 & 0 & L_{a_1} \\ 0 & 0 & L_{a_2} \\ \hline L_{a_3} & L_{a_4} & 0 \end{array} \right) \middle| L_{a_i} \in L_R; 1 \le i \le 4 \right\} \subseteq V$, be a super matrix vector subspace of refined labels over L_R of V.

Now as in case of super row vector space of refined labels we can in the case of general super $m \times n$ matrix of refined label of vector space define the notion of direct sum of vector subspaces and pseudo direct sum of super vector subspaces. This task is simple and hence is left as an exercise to the reader.

However we illustrate this situation by some examples.

Example 4.44: Let V =

$$\left\{ \left[\begin{array}{c|cc} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array} \right] \middle| L_{a_i} \in L_R; 1 \le i \le 12 \right\}$$ be a super matrix vector space of refined labels over the field L_R. Consider W_1 =

$$\left\{ \left(\begin{array}{c|cc} L_{a_1} & 0 & 0 \\ L_{a_2} & 0 & 0 \\ \hline 0 & L_{a_3} & 0 \\ 0 & 0 & 0 \end{array} \right) \middle| L_{a_i} \in L_R; 1 \le i \le 3 \right\} \subseteq V,$$ be a super matrix row vector subspace of V over L_R.

$$W_2 = \left\{ \left(\begin{array}{c|cc} 0 & L_{a_3} & 0 \\ 0 & 0 & 0 \\ \hline L_{a_1} & 0 & 0 \\ L_{a_2} & 0 & 0 \end{array} \right) \middle| L_{a_i} \in L_R; 1 \le i \le 3 \right\} \subseteq V,$$

$$W_3 = \left\{ \left(\begin{array}{c|cc} 0 & 0 & L_{a_1} \\ 0 & 0 & L_{a_2} \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 2 \right\} \subseteq V,$$

$$W_4 = \left\{ \left(\begin{array}{c|cc} 0 & 0 & 0 \\ 0 & L_{a_1} & 0 \\ \hline 0 & 0 & L_{a_2} \\ 0 & 0 & 0 \end{array} \right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 2 \right\} \subseteq V \text{ and}$$

$$W_5 = \left\{ \left(\begin{array}{c|cc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & L_{a_1} & L_{a_2} \end{array} \right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 2 \right\} \subseteq V, \text{ be super matrix}$$

vector subspaces of V over L_R. We see clearly V = $\bigcup_{i=1}^{5} W_i$ and

$$W_i \cap W_j = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right)$$ if $i \neq j$, $1 \leq i, j \leq 5$. Thus V is the direct sum of super matrix vector subspaces of V over L_R.

Example 4.45: Let V =

$$\left\{ \left(\begin{array}{c|c|cc|ccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} \\ L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} \\ L_{a_{15}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \end{array} \right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 21 \right\}$$

be a super row vector, vector space of refined labels over L_R.

Consider

$$W_1 = \left\{ \left(\begin{array}{c|c|cc|ccc} L_{a_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ L_{a_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ L_{a_3} & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 3 \right\} \subseteq V,$$

$$W_2 = \left\{ \left(\begin{array}{c|c|cc|ccc} 0 & L_{a_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & L_{a_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & L_{a_3} & 0 & 0 & 0 & 0 & 0 \end{array} \right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 3 \right\} \subseteq V,$$

$$W_3 = \left\{ \left(\begin{array}{c|c|cc|ccc} 0 & 0 & L_{a_1} & L_{a_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & L_{a_3} & L_{a_4} & 0 & 0 & 0 \end{array} \right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 4 \right\} \subseteq V,$$

$$W_4 = \left\{ \left(\begin{array}{c|c|cc|ccc} 0 & 0 & 0 & 0 & L_{a_3} & 0 & 0 \\ 0 & 0 & L_{a_1} & L_{a_2} & L_{a_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & L_{a_5} & 0 & 0 \end{array} \right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 5 \right\} \subseteq V,$$

$$W_5 = \left\{ \left(\begin{array}{c|c|cc|ccc} 0 & 0 & 0 & 0 & 0 & L_{a_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & L_{a_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & L_{a_3} & 0 \end{array} \right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 3 \right\} \subseteq V \text{ and}$$

$$W_6 = \left\{ \left(\begin{array}{c|c|cc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & L_{a_1} \\ 0 & 0 & 0 & 0 & 0 & 0 & L_{a_2} \\ 0 & 0 & 0 & 0 & 0 & 0 & L_{a_3} \end{array} \right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 3 \right\} \subseteq V \text{ be}$$

super row vector, vector subspaces of V over the field L_R.

Clearly $V = \bigcup_{i=1}^{6} W_i$ and $W_i \cap W_j =$

$$\left(\begin{array}{c|c|cc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

; if $i \neq j$. Thus we see V is a direct sum of super row vector subspace of V over LR.

Example 4.46: Let V =

$$\left\{ \left(\begin{array}{c|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array}\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 16 \right\}$$

be a super matrix vector space of refined labels over L_R.

Consider $W_1 = \left\{ \left(\begin{array}{c|cc|c} L_{a_1} & 0 & 0 & 0 \\ L_{a_2} & 0 & 0 & 0 \\ L_{a_3} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 3 \right\} \subseteq V,$

$W_2 = \left\{ \left(\begin{array}{c|cc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & L_{a_2} & 0 & 0 \\ \hline L_{a_1} & L_{a_3} & 0 & L_{a_4} \end{array}\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 4 \right\} \subseteq V,$

$W_3 = \left\{ \left(\begin{array}{c|cc|c} 0 & 0 & 0 & L_{a_2} \\ 0 & 0 & 0 & L_{a_3} \\ 0 & 0 & 0 & L_{a_4} \\ \hline 0 & 0 & L_{a_1} & 0 \end{array}\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 4 \right\} \subseteq V;$

$$W_4 = \left\{ \left(\begin{array}{c|cc|c} 0 & L_{a_1} & 0 & 0 \\ 0 & L_{a_2} & 0 & 0 \\ 0 & 0 & L_{a_3} & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right) \middle| L_{a_i} \in L_R; 1 \le i \le 3 \right\} \subseteq V \text{ and}$$

$$W_5 = \left\{ \left(\begin{array}{c|cc|c} 0 & 0 & L_{a_1} & 0 \\ 0 & 0 & L_{a_2} & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right) \middle| L_{a_i} \in L_R; 1 \le i \le 3 \right\} \subseteq V$$

be super matrix vector subspaces of V over the refined field L_R.

$$\text{Clearly } V = \bigcup_{i=1}^{5} W_i \text{ and } W_i \cap W_j = \left(\begin{array}{c|cc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right) \text{if } i \ne j,$$

$1 \le i, j \le 5$. Thus V is the direct sum of super vector subspaces of V over L_R.

Now having seen the concept of direct sum we now proceed onto give examples of pseudo direct sum of subspaces.

Example 4.47: Let V =

$$\left\{ \left(\begin{array}{c|c|ccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \end{array} \right) \middle| L_{a_i} \in L_R; 1 \le i \le 15 \right\} \text{be a super}$$

row vector, vector space of refined labels over L_R.

Consider

$$W_1 = \left\{\left(\begin{array}{c|c|ccc} L_{a_1} & 0 & 0 & 0 & 0 \\ L_{a_2} & 0 & 0 & 0 & 0 \\ L_{a_3} & 0 & L_{a_4} & 0 & L_{a_5} \end{array}\right) \middle| L_{a_i} \in L_R; 1 \le i \le 5\right\} \subseteq V,$$

$$W_2 = \left\{\left(\begin{array}{c|c|ccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & L_{a_3} & 0 & 0 \\ L_{a_1} & L_{a_2} & 0 & 0 & L_{a_4} \end{array}\right) \middle| L_{a_i} \in L_R; 1 \le i \le 4\right\} \subseteq V,$$

$$W_3 = \left\{\left(\begin{array}{c|c|ccc} 0 & L_{a_2} & 0 & 0 & 0 \\ 0 & L_{a_3} & 0 & 0 & 0 \\ L_{a_1} & L_{a_4} & 0 & 0 & 0 \end{array}\right) \middle| L_{a_i} \in L_R; 1 \le i \le 4\right\} \subseteq V,$$

$$W_4 = \left\{\left(\begin{array}{c|c|ccc} 0 & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L_{a_6} \end{array}\right) \middle| L_{a_i} \in L_R; 1 \le i \le 6\right\} \subseteq V \text{ and}$$

$$W_5 = \left\{\left(\begin{array}{c|c|ccc} 0 & L_{a_2} & L_{a_3} & 0 & 0 \\ 0 & 0 & L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_1} & 0 & L_{a_7} & L_{a_8} & L_{a_9} \end{array}\right) \middle| L_{a_i} \in L_R; 1 \le i \le 9\right\} \subseteq V \text{ are}$$

super row vector subspaces of V over L_R.

We see clearly $V = \bigcup_{i=1}^{5} W_i$ and $W_i \cap W_j \neq$

$$\left(\begin{array}{c|c|ccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \text{if } i \neq j,\ 1 \le i, j \le 5.$$

Thus V is not the direct sum of super row vector subspaces of V but only a pseudo direct sum of super row vector subspaces of V over L_R.

Example 4.49: Let V =

$$\left\{\left(\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ \hline L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ \hline L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \\ L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ L_{a_{25}} & L_{a_{26}} & L_{a_{27}} \end{array}\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 27 \right\}$$

be a super column vector, vector space of refined labels over L_R.

$$\text{Consider } W_1 = \left\{\left(\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ 0 & 0 & 0 \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 9 \right\} \subseteq V,$$

$$W_2 = \left\{ \left(\begin{array}{ccc} L_{a_1} & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & L_{a_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ 0 & 0 & 0 \\ L_{a_7} & L_{a_8} & L_{a_9} \end{array} \right) \middle| L_{a_i} \in L_R ; 1 \le i \le 9 \right\} \subseteq V,$$

$$W_3 = \left\{ \left(\begin{array}{ccc} L_{a_1} & 0 & 0 \\ L_{a_2} & L_{a_3} & L_{a_4} \\ \hline 0 & L_{a_5} & 0 \\ L_{a_6} & L_{a_7} & L_{a_8} \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \middle| L_{a_i} \in L_R ; 1 \le i \le 8 \right\} \subseteq V,$$

$$W_4 = \left\{ \left(\begin{array}{ccc} L_{a_1} & 0 & 0 \\ 0 & 0 & L_{a_2} \\ \hline 0 & 0 & 0 \\ 0 & 0 & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & L_{a_7} \end{array} \right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 7 \right\} \subseteq V \text{ and}$$

$$W_5 = \left\{ \left(\begin{array}{ccc} L_{a_1} & 0 & 0 \\ 0 & 0 & L_{a_2} \\ \hline L_{a_3} & 0 & 0 \\ 0 & L_{a_4} & 0 \\ 0 & 0 & L_{a_5} \\ \hline 0 & L_{a_6} & 0 \\ \hline L_{a_7} & 0 & 0 \\ L_{a_8} & L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & 0 & 0 \end{array} \right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 11 \right\} \subseteq V \text{ are super}$$

column vector, vector subspaces of V over L_R.

Clearly $V = \bigcup_{i=1}^{5} W_i$ and $W_i \cap W_j \neq \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ if $i \neq j$, $1 \leq i$, $j \leq 5$.

Thus V is only a pseudo direct sum of super column vector, vector subspaces of V over L_R.

Example 4.49: Let V =

$$\left\{\left(\begin{array}{c|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ \hline L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \\ L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \\ \hline L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \end{array}\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 24 \right\}$$ be a super

matrix vector space of refined labels over L_R. Consider W_1 =

$$\left\{\left(\begin{array}{c|cc|c} L_{a_1} & 0 & 0 & L_{a_3} \\ L_{a_2} & 0 & 0 & L_{a_4} \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 4 \right\} \subseteq V,$$

$$W_2 = \left\{ \left(\begin{array}{c|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & 0 \\ 0 & L_{a_4} & L_{a_5} & L_{a_6} \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right) \middle| L_{a_i} \in L_R; 1 \le i \le 6 \right\} \subseteq V,$$

$$W_3 = \left\{ \left(\begin{array}{c|cc|c} 0 & L_{a_2} & 0 & L_{a_4} \\ L_{a_1} & 0 & L_{a_3} & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline L_{a_5} & 0 & L_{a_6} & L_{a_7} \end{array} \right) \middle| L_{a_i} \in L_R; 1 \le i \le 7 \right\} \subseteq V,$$

$$W_4 = \left\{ \left(\begin{array}{c|cc|c} L_{a_1} & L_{a_2} & 0 & L_{a_3} \\ 0 & 0 & 0 & 0 \\ \hline L_{a_4} & 0 & 0 & L_{a_7} \\ L_{a_5} & 0 & 0 & L_{a_8} \\ L_{a_6} & 0 & 0 & L_{a_9} \\ \hline 0 & 0 & 0 & 0 \end{array} \right) \middle| L_{a_i} \in L_R; 1 \le i \le 9 \right\} \subseteq V,$$

$$W_5 = \left\{ \left(\begin{array}{c|cc|c} L_{a_1} & 0 & 0 & 0 \\ 0 & 0 & L_{a_9} & L_{a_{10}} \\ \hline L_{a_2} & L_{a_4} & 0 & 0 \\ 0 & L_{a_5} & 0 & 0 \\ 0 & L_{a_6} & 0 & 0 \\ \hline L_{a_3} & 0 & L_{a_7} & L_{a_8} \end{array} \right) \middle| L_{a_i} \in L_R; 1 \le i \le 10 \right\} \subseteq V, \text{ and}$$

$$W_6 = \left\{ \left(\begin{array}{c|cc|c} L_{a_1} & 0 & 0 & 0 \\ 0 & 0 & L_{a_4} & 0 \\ \hline 0 & 0 & L_{a_6} & 0 \\ L_{a_2} & 0 & L_{a_7} & 0 \\ 0 & 0 & L_{a_8} & L_{a_9} \\ \hline L_{a_5} & L_{a_3} & 0 & 0 \end{array} \right) \middle| L_{a_i} \in L_R ; 1 \leq i \leq 9 \right\} \subseteq V, \text{ be super}$$

vector subspace of V over L_R. V = $\bigcup_{i=1}^{6} W_i$ and $W_i \cap W_j \neq$

$$\left[\begin{array}{c|cc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right] \text{if } i \neq j,\ 1 \leq i, j \leq 6.$$

Thus V is a pseudo direct sum of super matrix vector subspaces of V over L_R.

Now we can as in case of vector spaces define linear transformation, linear operator and projection. This task is left as an exercise to the reader. Also all super matrix vector spaces of refined labels over L_R.

Example 4.50: Let V and W be two super matrix vector spaces of refined labels over L_R, where V =

$$\left\{ \left(\begin{array}{cc|c|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array} \right) \middle| L_{a_i} \in L_R ; 1 \leq i \leq 16 \right\} \text{ be a super matrix}$$

vector space of refined labels over L_R.

Consider W =

$$\left\{ \left(\begin{array}{cc|ccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ \hline L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ \hline L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \end{array} \right) \middle| L_{a_i} \in L_R ; 1 \leq i \leq 15 \right\}$$

be a super matrix vector space of refined labels of L_R.

Define T : V → W by

$$T \left(\left[\begin{array}{cc|c|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array} \right] \right) = \left[\begin{array}{cc|ccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ \hline L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ \hline L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \end{array} \right]$$

Clearly T is a linear transformation of super matrix vector space of refined labels over L_R.

Example 4.51: Let V =

$$\left\{ \left(\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \end{array} \right) \middle| L_{a_i} \in L_R ; 1 \leq i \leq 15 \right\}$$ be a super matrix vector space of refined labels over L_R.

Consider T : V → V a map defined by

$$T \left(\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \end{array} \right) = \left(\begin{array}{cc|c} L_{a_1} & L_{a_2} & 0 \\ 0 & 0 & L_{a_3} \\ \hline 0 & L_{a_4} & L_{a_6} \\ L_{a_5} & 0 & 0 \\ \hline 0 & L_{a_7} & L_{a_8} \end{array} \right)$$

Clearly T is a linear operator on V.

Now consider W = $\left\{ \left(\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 9 \right\}$

$\subseteq$ V, W is a super matrix vector subspace of V of refined labels over L_R.

$$P\left(\left[\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \end{array} \right] \right) = \left(\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right).$$

It is easily verified that P is a projection of V to W and P (W) $\subseteq$ W; with $P^2 = P$ on V.

We can as in case of super row vector space derive all the related theorems given in this chapter with appropriate modifications and this work can be carried out by the reader.

However we repeatedly make a mention that super matrix linear algebra of refined labels cannot be constructed as it is not possible to get any collection of multiplicative wise compatible super matrices of labels.

Now we proceed onto define the notion of linear functional of refined labels L_R.

Here once again we call that the refined labels L_R is isomorphic with R. So we can take the refined labels itself as a field or R as a field.

Let V be a super matrix vector space of refined labels over L_R. Define a linear transformation from V into the field of refined labels called the linear functional of refined labels on V.

If $f : V \to L_R$ then
$f(L_c\, \alpha + \beta) = L_c\, f(\alpha) + f(\beta)$ where $\alpha, \beta \in V$ and $L_c \in L_R$.

We will first illustrate this concept by some examples.

Example 4.52: Let V = $\left\{ \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ L_{a_4} \\ L_{a_5} \\ \hline L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \end{array}\right] \middle| L_{a_i} \in L_R; 1 \le i \le 8 \right\}$ be a super column vector space of refined labels over the field of refined labels L_R.

$$T = \left(\left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ L_{a_4} \\ L_{a_5} \\ \hline L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \end{array}\right] \right) = L_{a_1} + \ldots + L_{a_8}$$

$$= L_{a_1 + a_2 + \ldots + a_8} \in L_R.$$

T is a linear functional on V.

Infact the concept of linear functional can lead to several interesting applications.

Example 4.53: Let V = $\left\{\left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ L_{a_6} \\ L_{a_7} \end{array}\right] \middle| L_{a_i} \in L_R ; 1 \leq i \leq 7\right\}$ be a super column vector space of refined labels over L_R.

Define f : V $\to$ L_R by

$$f\left(\left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ L_{a_6} \\ L_{a_7} \end{array}\right]\right) = L_{a_1} + L_{a_2} + L_{a_3}$$

= $L_{a_1+a_2+a_3}$ $\in$ L_R. f is a linear functional on V.

Consider f_1 : V $\to$ L_R by

$$f_1\left(\begin{bmatrix} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ L_{a_6} \\ L_{a_7} \end{bmatrix}\right) = L_{a_4}$$

; f_1 is also a linear functional on V. We can define several linear functional on V.

Consider $f_2\left(\begin{bmatrix} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ L_{a_6} \\ L_{a_7} \end{bmatrix}\right) = L_{a_1} + L_{a_4} + L_{a_7}$

$= L_{a_1+a_4+a_7} \in L_R$;

f_2 is a linear functional on V. We see all the three linear functionals f_1, f_2 and f are distinct.

Example 4.54: Let V =

$$\left\{\left(L_{a_1} \mid L_{a_2} \quad L_{a_3} \mid L_{a_4} \quad L_{a_5} \quad L_{a_6} \mid L_{a_7} \quad L_{a_8}\right) \middle| L_{a_i} \in L_R ; 1 \leq i \leq 8\right\}$$

be a super row vector space of refined labels over L_R.

Define $f : V \to L_R$ by

$f\left(\left(L_{a_1} \mid L_{a_2} \quad L_{a_3} \mid L_{a_4} \quad L_{a_5} \quad L_{a_6} \mid L_{a_7} \quad L_{a_8}\right)\right)$

$= L_{a_1} + L_{a_2} + L_{a_4} + L_{a_7}$

$= L_{a_1+a_2+a_4+a_7}$; f is a linear functional on V.

We can define several linear functional on V.

Example 4.55: Let V =

$$\left\{\left.\begin{bmatrix} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ \hline L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \\ L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ L_{a_{25}} & L_{a_{26}} & L_{a_{27}} \\ L_{a_{28}} & L_{a_{29}} & L_{a_{30}} \end{bmatrix} \right| L_{a_i} \in L_R; 1 \leq i \leq 30 \right\}$$

be a super column vector, vector space of refined labels over L_R, the field of refined labels.

Define $f : V \to L_R$ by

$$f\left(\begin{bmatrix} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ \hline L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \\ L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ L_{a_{25}} & L_{a_{26}} & L_{a_{27}} \\ L_{a_{28}} & L_{a_{29}} & L_{a_{30}} \end{bmatrix}\right)$$

$= L_{a_1} + L_{a_5} + L_{a_7} + L_{a_{11}} + L_{a_{13}} + L_{a_{17}} + L_{a_{19}} + L_{a_{23}} + L_{a_{29}}$

$= L_{a_1+a_5+a_7+a_{11}+a_{13}+a_{17}+a_{19}+a_{23}+a_{29}}$, f is a linear functional on V.

Example 4.56: Let V =

$$\left\{ \left[\begin{array}{c|ccc|cc} L_{a_1} & L_{a_5} & L_{a_9} & L_{a_{13}} & L_{a_{17}} & L_{a_{21}} \\ L_{a_2} & L_{a_6} & L_{a_{10}} & L_{a_{14}} & L_{a_{18}} & L_{a_{22}} \\ L_{a_3} & L_{a_7} & L_{a_{11}} & L_{a_{15}} & L_{a_{19}} & L_{a_{23}} \\ L_{a_4} & L_{a_8} & L_{a_{12}} & L_{a_{16}} & L_{a_{20}} & L_{a_{24}} \end{array}\right] \middle| L_{a_i} \in L_R; 1 \le i \le 24 \right\}$$ be a

super row vector space of refined labels over L_R.

Consider $f : V \to L_R$ by

$$f\left(\left[\begin{array}{c|ccc|cc} L_{a_1} & L_{a_5} & L_{a_9} & L_{a_{13}} & L_{a_{17}} & L_{a_{21}} \\ L_{a_2} & L_{a_6} & L_{a_{10}} & L_{a_{14}} & L_{a_{18}} & L_{a_{22}} \\ L_{a_3} & L_{a_7} & L_{a_{11}} & L_{a_{15}} & L_{a_{19}} & L_{a_{23}} \\ L_{a_4} & L_{a_8} & L_{a_{12}} & L_{a_{16}} & L_{a_{20}} & L_{a_{24}} \end{array}\right] \right) = L_{a_1} + L_{a_6} + L_{a_{11}} + L_{a_{24}}$$

$= L_{a_1+a_6+a_{11}+a_{24}}$ is a linear functional on V.

Example 4.57: Let V =

$$\left\{ \left[\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \end{array}\right] \middle| L_{a_i} \in L_R; 1 \le i \le 15 \right\}$$ be a super matrix

super vector space of refined labels over L_R, the refined field of labels.

Define $f : V \to L_R$ by

$$f\left(\left[\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \end{array}\right] \right)$$

$= L_{a_1} + L_{a_5} + L_{a_7} + L_{a_{11}} + L_{a_6} + L_{a_{12}} + L_{a_{13}}$

$= L_{a_1+a_5+a_7+a_{11}+a_6+a_{12}+a_{13}}$;

f is a linear functional on V.

We as in case of vector space define the collection of all linear functionals on V, V a super matrix vector space over L_R as the dual space and it is denoted by L (V, L_R).

Here we define $L_{\delta_{ij}} = 0$ if $i \neq j$ if $i = j$ then $L_{\delta_{ij}} = L_{m+1}$.

Using this convention we can prove all results analogous to L (V, F), F any field.

For instance if V =
$\left\{\left(L_{a_1} \mid L_{a_2} \mid L_{a_3} \quad L_{a_4}\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 4\right\}$ be a super row vector space of refined labels over L_R then B = {(L_{m+1} | 0 | 0 0) = v_1, v_2 = (0 | L_{m+1} | 0 0), v_3 = (0 | 0 | L_{m+1} | 0), v_4 = (0 | 0 | 0 L_{m+1})} $\subseteq$ V is a basis of V over L_R.

Define $f_i(v_j) = L_{\delta_{ij}}$; $1 \leq i, j \leq 4$; (f_1, f_2, f_3, f_4) is a basis of L (V, L_R) over L_R.

We as in case of usual vector spaces define if f : V $\rightarrow$ L_R is a linear functional on super vector space of refined labels with appropriate modifications then null space of f is of dimension n – 1.

It is however pertinent to mention here that we may have several super vector spaces of dimension n which need not be isomorphic as in case of usual vector spaces with this in mind we can say or derive results with suitable modifications.

However for any super matrix vector space V of refined labels over L_R the refined hyper super space of V as a super subspace of dimension n – 1 where n is the dimension of V.

Consider V = $\left\{\left(\begin{array}{c|c} L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \end{array}\right) \middle| L_{a_i} \in L_R; 1 \le i \le 4\right\}$ be a super matrix vector space of dimension four over the refined field L_R.

Consider W = $\left\{\left(\begin{array}{c|c} L_{a_1} & L_{a_2} \\ 0 & L_{a_3} \end{array}\right) \middle| L_{a_i} \in L_R; 1 \le i \le 3\right\} \subseteq$ V, W is the hyper super space of V and dimension of W is three over L_R.

Example 4.58: Let V = $\left\{\left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ L_{a_4} \\ \hline L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ \hline L_{a_8} \\ L_{a_9} \\ \hline L_{a_{10}} \end{array}\right] \middle| L_{a_i} \in L_R; 1 \le i \le 10\right\}$ be a super column vector space of dimension 10 over the field L_R of refined labels.

Consider M = $\left\{ \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ L_{a_4} \\ \hline 0 \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \\ \hline L_{a_9} \end{array}\right] \middle| L_{a_i} \in L_R; 1 \le i \le 9 \right\}$ $\subseteq$ V is a hyper super space of refined labels of V over L_R. Infact V has several hyper super spaces.

Example 4.59: Let V =

$$\left\{ \left[\begin{array}{c|cc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \end{array}\right] \middle| L_{a_i} \in L_R; 1 \le i \le 18 \right\}$$ be a super matrix vector space of refined labels over L_R.

Consider W = $\left\{ \left[\begin{array}{c|cc} 0 & L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_6} & L_{a_7} & L_{a_8} \\ \hline L_{a_9} & L_{a_{10}} & L_{a_{11}} \\ L_{a_{12}} & L_{a_{13}} & L_{a_{14}} \\ L_{a_{15}} & L_{a_{16}} & L_{a_{17}} \end{array}\right] \middle| L_{a_i} \in L_R; 1 \le i \le 17 \right\} \subseteq V,$

be a super matrix vector subspace of V over L_R.

Clearly W is a hyper super space of refined labels over L_R of V. Dimension of V is 18 where as dimension of W over L_R is 17.

Let V be a super vector space of refined labels over L_R. S be a subset of V, the annihilator of S is the set S^o of all super linear functionals f on V such that f (α) = 0 for every $\alpha \in$ S.

Let V = $\left\{\left[L_{a_1} \mid L_{a_2} \mid L_{a_3}\right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 3\right\}$ be a super row vector space of refined labels over L_R. We will only make a trial.

Suppose W = $\left\{\left[L_{a_1} \mid 0 \mid 0\right] \middle| L_{a_i} \in L_R\right\} \subseteq$ V be a subset of V.

Define f : V $\to$ V by

$f\left(\left(L_{a_1} \mid 0 \mid 0\right)\right) = \left(0 \mid 0 \mid 0\right)$ every $\left(L_{a_1} \mid 0 \mid 0\right) \in$ W.

$W^o = \{f \in L(V, L_R) \mid f\left(\left(L_{a_1} \mid 0 \mid 0\right)\right)$

$= \left(0 \mid 0 \mid 0\right)$ for every $L_{a_1} \in L_R\}$.

We see W^o is a subspace of L (V, L_R) as it contains only f and f_o the zero function. However one has to study in this direction to get analogous results.

We will give some more examples.

Consider K =

$$\left\{\left[\begin{array}{cc|ccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ \hline L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \end{array}\right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 20\right\}$$

be a super matrix vector space of refined labels over the field L_R.

Let S = $\left\{\left[\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & L_{a_1} & 0 & 0 & 0 \\ L_{a_2} & 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|ccc} L_{a_1} & L_{a_2} & 0 & 0 & 0 \\ L_{a_3} & L_{a_4} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],\right.$

$\left.\left[\begin{array}{cc|ccc} 0 & 0 & L_{a_1} & L_{a_2} & L_{a_3} \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]\right\}$ where $L_{a_i} \in L_R$; $1 \leq i \leq 4\} \subseteq V$ be a proper subset of V. Clearly S is not a super vector subspace of V it is only a proper subset of V.

Define $f_i : V \to V$ as follows:

$$f_1\left(\left[\begin{array}{cc|ccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ \hline L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \end{array}\right]\right)$$

$$= \left[\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} \\ \hline L_{a_5} & 0 & L_{a_{15}} & L_{a_{16}} & L_{a_{17}} \\ 0 & L_{a_8} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \end{array}\right],$$

$$f_2\left(\left[\begin{array}{cc|ccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ \hline L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \end{array}\right]\right) = \left[\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & L_{a_8} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$f_3\left(\left[\begin{array}{cc|ccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ \hline L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \end{array}\right]\right)$$

$$= \left[\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & L_{a_{18}} & L_{a_{19}} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$f_4\left(\left[\begin{array}{cc|ccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ \hline L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \end{array}\right]\right) =$$

$$\left[\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & L_{a_1} & L_{a_2} & L_{a_3} \\ 0 & 0 & L_{a_4} & L_{a_5} & L_{a_6} \end{array}\right] \text{ and so on.}$$

We see f_1, f_2, f_3 and f_4 are defined such that f_i (x) = $\left(\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$ for every $x \in S$, $1 \leq i \leq 4$. Now having seen how S^o looks like we can give a subspace structure of L (V, L_R).

We can as in case of usual vector space derive properties related with the dual super space and properties enjoyed by the super linear functionals or linear functionals on super matrix vector spaces of refined labels. The only criteria being L_R is isomorphic to R. Also we have the properties used in the super linear algebra regarding super vector spaces [47].

Chapter Five

SUPER SEMIVECTOR SPACES OF REFINED LABELS

In this chapter we for the first time introduce the notion of super semivector space of refined labels.

We have defined the notion of semigroup of refined labels. Also we have seen the set of labels $L_a \in L_{R^+ \cup \{0\}}$ forms a semifield of refined labels. Also $L_{Q^+ \cup \{0\}}$ is again a semifield of refined labels. We would be using the semifield of refined labels to build super semivector spaces of different types.

Let

$$M = \left\{ \left(L_{a_1} \quad L_{a_2} \mid L_{a_3} \mid L_{a_4} \quad L_{a_5} \quad L_{a_6} \mid \ldots \mid L_{a_{n-1}} \quad L_{a_n} \right) \right|$$

$L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq n \Big\}$ be the collection of all super row vectors of refined labels.

Clearly M is a semigroup under addition. We see M is infact only an infinite commutative semigroup under the operation of addition. Clearly M is not a group under addition. Further we cannot define product on M as it is not compatible

under any form of product being a super row vector. We know $L_{R^+ \cup \{0\}}$ and $L_{Q^+ \cup \{0\}}$ are semifields. Now we can define super semivector space of refined labels.

DEFINITION 5.1: *Let*

$$V = \{(L_{a_1} \ \ L_{a_2} \mid L_{a_3} \mid L_{a_4} \ \ L_{a_5} \ \ L_{a_6} \mid \ldots \mid L_{a_{n-1}} \ \ L_{a_n}) \big| L_{a_i} \in L_{R^+ \cup \{0\}};$$

$1 \leq i \leq n\}$ be a semigroup under addition with $(L_0 \ \ L_0 \mid L_0 \mid L_0 \ \ L_0 \ \ L_0 \mid L_0 \mid L_0 \ \ L_0)$ as the zero row vector of refined labels. V is a super semivector space of refined labels over $L_{R^+ \cup \{0\}}$; the semifield of refined labels isomorphic with $R^+ \cup \{0\}$.

We will first illustrate this situation by some simple examples.

Example 5.1: Let

$$V = \{(L_{a_1} \ \ L_{a_2} \mid L_{a_3} \mid L_{a_4}) \big| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 4\}$$

be a super semivector space of refined labels over the semifield $L_{R^+ \cup \{0\}}$.

Example 5.2: Let

$$V = \{(L_{a_1} \ \ L_{a_2} \ \ L_{a_3} \mid L_{a_4} \ \ L_{a_5} \mid L_{a_6} \mid L_{a_7} \ \ L_{a_8} \ \ L_{a_9} \ \ L_{a_{10}}) \big|$$

$L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 10\}$ be a super semivector space of refined labels over the semifield $L_{R^+ \cup \{0\}}$. We also call V to be a super row semivector space of refined labels. Since from the very context one can easily understand we do not usually qualify these spaces by row or column.

Example 5.3: Let

$$S = \{(L_{a_1} \ \ L_{a_2} \ \ L_{a_3} \mid L_{a_4} \mid L_{a_5} \mid L_{a_6} \mid L_{a_7}) \big| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 7\}$$

be a super semivector space of refined labels over the semifield $L_{R^+ \cup \{0\}}$.

Example 5.4: Let

$$M = \left\{\left(L_{a_1} \mid L_{a_2} \quad L_{a_3} \mid L_{a_4} \mid L_{a_5} \quad L_{a_6}\right) \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 6\right\}$$

be a super semivector space of refined labels over the semifield $L_{Q^+ \cup \{0\}}$.

Example 5.5: Let

$$V = \left\{\left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \mid L_{a_4} \quad L_{a_5} \mid L_{a_6} \quad L_{a_7} \quad L_{a_8} \quad L_{a_9}\right) \middle|\right.$$

$\left. L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 9\right\}$ be a super semivector space of refined labels over $L_{Q^+ \cup \{0\}}$ the semifield of refined labels isomorphic with $Q^+ \cup \{0\}$.

Example 5.6: Let

$$K = \left\{\left(L_{a_1} \quad L_{a_2} \mid L_{a_3} \quad L_{a_4} \quad L_{a_5} \mid L_{a_6} \quad L_{a_7} \mid L_{a_8}\right) \middle| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \leq i \leq 8\right\}$$

be a super semivector space of refined labels over $L_{Q^+ \cup \{0\}}$. Clearly K is not a super semivector space of refined labels over $L_{R^+ \cup \{0\}}$.

Example 5.7: Let

$$V = \left\{\left(L_{a_1} \quad L_{a_2} \mid L_{a_3} \quad L_{a_4} \quad L_{a_5} \quad L_{a_6} \mid L_{a_7} \mid L_{a_8} \quad L_{a_9} \quad L_{a_{10}}\right) \middle|\right.$$

$\left. L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \leq i \leq 10\right\}$ be a super semivector space of refined labels over $L_{Q^+ \cup \{0\}}$. It is pertinent to mention here that it does not make any difference whether we define the super semivector space over $R^+ \cup \{0\}$ or $L_{R^+ \cup \{0\}}$ (or $Q^+ \cup \{0\}$ or $L_{Q^+ \cup \{0\}}$) as $R^+ \cup \{0\}$ is isomorphic to $L_{R^+ \cup \{0\}}$ ($Q^+ \cup \{0\}$ is isomorphic with $L_{Q^+ \cup \{0\}}$).

We can as in case of semivector spaces define substructures on them. This is simple and so left as an exercise to the reader. We give some simple examples of substructures and illustrate their properties also with examples.

Example 5.8: Let

$$V = \left\{\left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \mid L_{a_4} \quad L_{a_5} \mid L_{a_6} \quad L_{a_7} \quad L_{a_8} \quad L_{a_9}\right)\right|$$

$L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 9\Big\}$ be a super semivector space of refined labels over $L_{R^+ \cup \{0\}}$. Consider

$$M = \left\{\left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \mid 0 \quad 0 \mid L_{a_4} \quad 0 \quad L_{a_5} \quad 0\right)\right|$$

$L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 5\Big\} \subseteq$ V, M is a super semivector subspace of V over the semifield $L_{R^+ \cup \{0\}}$.

Example 5.9: Let

$$V = \left\{\left(L_{a_1} \mid L_{a_2} \quad L_{a_3} \quad L_{a_4} \quad L_{a_5} \mid L_{a_6} \quad L_{a_7} \quad L_{a_8} \mid L_{a_9} \quad L_{a_{10}} \mid L_{a_{11}}\right)\right|$$

$L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 11\Big\}$ be a super semivector space of refined labels over the semifield $L_{R^+ \cup \{0\}}$. Consider

$$W = \left\{\left(L_{a_1} \mid L_{a_2} \quad L_{a_3} \quad L_{a_4} \quad L_{a_5} \mid L_{a_6} \quad 0 \quad 0 \mid 0 \quad 0 \mid 0\right)\right|$$

$L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 6\Big\} \subseteq$ V, is a super semivector subspace of V over the refined label semifield $L_{R^+ \cup \{0\}}$.

Example 5.10: Let

$$V = \left\{\left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \quad L_{a_4} \mid L_{a_5} \quad L_{a_6} \mid L_{a_7} \quad L_{a_8} \quad L_{a_9} \quad L_{a_{10}} \mid L_{a_{11}} \mid L_{a_{12}}\right)\right|$$

$L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 12\Big\}$ be a super semivector space of refined labels over the semifield $L_{Q^+ \cup \{0\}}$.

Consider

$$W = \left\{\left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \quad L_{a_4} \mid L_{a_5} \quad L_{a_6} \mid L_{a_7} \quad L_{a_8} \quad L_{a_9} \quad L_{a_{10}} \mid L_{a_{11}} \mid L_{a_{12}}\right)\right|$$

$L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \leq i \leq 12\Big\} \subseteq$ V; W is a super semivector subspace of V of refined labels over the semifield $L_{Q^+ \cup \{0\}}$.

Example 5.11: Let

$$V = \left\{\left(L_{a_1} \quad L_{a_2} \mid L_{a_3} \quad L_{a_4} \quad L_{a_5} \mid L_{a_6}\right) \middle| L_{a_i} \in L_{Q^+\cup\{0\}}; 1 \le i \le 6\right\}$$

be a super semivector space of refined labels over the semifield $L_{Q^+\cup\{0\}}$.

Consider

$$M = \left\{\left(L_{a_1} \quad 0 \mid L_{a_2} \quad 0 \quad L_{a_3} \mid 0\right) \middle| L_{a_i} \in L_{Q^+\cup\{0\}}; 1 \le i \le 3\right\}$$

$\subseteq$ V, is a super semivector subspace of V of refined labels over $L_{Q^+\cup\{0\}}$.

Now having seen super semivector subspaces of a super semivector space we can now proceed onto give examples of the notion of direct sum and pseudo direct sum of super semivector subspaces of a super semivector space.

Example 5.12: Let

$$V = \left\{\left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \mid L_{a_4} \quad L_{a_5} \mid L_{a_6}\right) \middle| L_{a_i} \in L_{R^+\cup\{0\}}; 1 \le i \le 6\right\}$$

be a super semivector space of refined labels over the semifield of refined labels $L_{R^+\cup\{0\}}$.

Consider

$$M_1 = \left\{\left(L_{a_1} \quad 0 \quad L_{a_2} \mid 0 \quad 0 \mid 0\right) \middle| L_{a_i} \in L_{R^+\cup\{0\}}; 1 \le i \le 2\right\} \subseteq V,$$

$$M_2 = \left\{\left(0 \quad L_{a_1} \quad 0 \mid 0 \quad 0 \mid L_{a_2}\right) \middle| L_{a_i} \in L_{R^+\cup\{0\}}; 1 \le i \le 2\right\} \subseteq V,$$

and

$$M_3 = \left\{\left(0 \quad 0 \quad 0 \mid L_{a_1} \quad L_{a_2} \mid 0\right) \middle| L_{a_i} \in L_{R^+\cup\{0\}}; 1 \le i \le 2\right\} \subseteq V$$

be three super semivector subspace of V of refined labels over $L_{R^+\cup\{0\}}$. Clearly $V = \bigcup_{i=1}^{3} M_i$ with $M_i \cap M_j$ = {(0 0 0 | 0 0 | 0)} if $i \neq j$, $1 \le i, j \le 3$. Thus V is a direct sum super semivector subspaces M_1, M_2 and M_3 of V.

Example 5.13: Let

$$V = \left\{\left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \quad L_{a_4} \mid L_{a_5} \mid L_{a_6} \quad L_{a_7} \mid L_{a_8} \quad L_{a_9} \quad L_{a_{10}}\right)\right|$$

$L_{a_i} \in L_{Q^+\cup\{0\}}; 1 \le i \le 10\Big\}$ be a super semivector space of refined labels over $L_{Q^+\cup\{0\}}$.

Take

$W_1 = \left\{\left(L_{a_1} \; 0 \; 0 \; 0 \mid L_{a_2} \mid 0 \; 0 \mid 0 \; 0 \; 0\right) \Big| L_{a_i} \in L_{Q^+\cup\{0\}}; 1 \le i \le 2\right\} \subseteq V$,

$W_2 = \left\{\left(0 \; 0 \; 0 \; 0 \mid 0 \mid L_{a_1} \; L_{a_2} \mid L_{a_3} \; 0 \; 0\right) \Big| L_{a_i} \in L_{Q^+\cup\{0\}}; 1 \le i \le 3\right\} \subseteq V$,

$W_3 = \left\{\left(0 \; L_{a_1} \; L_{a_2} \; L_{a_3} \mid 0 \mid 0 \; 0 \mid 0 \; 0 \; 0\right) \Big| L_{a_i} \in L_{Q^+\cup\{0\}}; 1 \le i \le 3\right\} \subseteq V$

and

$W_4 = \left\{\left(0 \; 0 \; 0 \; 0 \mid 0 \mid 0 \; 0 \mid 0 \; L_{a_1} \; L_{a_2}\right) \Big| L_{a_i} \in L_{Q^+\cup\{0\}}; 1 \le i \le 2\right\} \subseteq V$;

be super semivector subspaces of V over the refined label semifield $L_{Q^+\cup\{0\}}$. We have $V = \bigcup_{i=1}^{4} W_i$ with $W_i \cap W_j = \{(0\ 0\ 0\ 0 \mid 0 \mid 0\ 0 \mid 0\ 0\ 0)\}$ if $i \ne j$, $1 \le i, j \le 4$. Thus V is a direct sum super semivector subspaces W_1, W_2, W_3 and W_4.

Example 5.14: Let

$$V = \left\{\left(L_{a_1} \mid L_{a_2} \; L_{a_3} \mid L_{a_4} \; L_{a_5} \; L_{a_6} \mid \right.\right.$$

$$\left.\left. L_{a_7} \; L_{a_8} \; L_{a_9} \; L_{a_{10}} \mid L_{a_{11}} \; L_{a_{12}} \mid L_{a_{13}} \; L_{a_{14}}\right)\right|$$

$L_{a_i} \in L_{R^+\cup\{0\}}; 1 \le i \le 14\Big\}$ be a super semivector space of refined labels over the semifield of refined labels $L_{R^+\cup\{0\}}$.

Consider

$$W_1 = \left\{\left(L_{a_1} \mid 0 \; 0 \mid L_{a_2} \; L_{a_3} \; L_{a_4} \mid 0 \; 0 \; 0 \; 0 \mid 0 \; 0 \mid 0 \; L_{a_5}\right)\right|$$

$$\left. L_{a_i} \in L_{R^+\cup\{0\}}; 1 \le i \le 5\right\} \subseteq V,$$

$$W_2 = \left\{\left(0 \mid L_{a_1} \; L_{a_2} \mid L_{a_3} \; 0 \; L_{a_4} \mid 0 \; 0 \; L_{a_5} \; L_{a_6} \mid 0 \; 0 \mid 0 \; 0\right)\right|$$

$$L_{a_i} \in L_{R^+\cup\{0\}}; 1 \le i \le 6\Big\} \subseteq V,$$

$$W_3 = \Big\{\Big(L_{a_1} \,|\, 0 \;\; 0 \,|\, 0 \;\; L_{a_2} \;\; L_{a_3} \,|\, L_{a_4} \;\; 0 \;\; 0 \;\; L_{a_5} \,|\, 0 \;\; 0 \,|\, L_{a_7} \;\; L_{a_6}\Big)\Big|$$

$$L_{a_i} \in L_{R^+\cup\{0\}}; 1 \le i \le 7\Big\} \subseteq V,$$

$$W_4 = \Big\{\Big(L_{a_1} \,|\, L_{a_2} \;\; 0 \,|\, 0 \;\; 0 \;\; 0 \,|\, L_{a_3} \;\; L_{a_4} \;\; L_{a_5} \;\; L_{a_6} \,|\, L_{a_7} \;\; 0 \,|\, L_{a_7} \;\; 0\Big)\Big|$$

$$L_{a_i} \in L_{R^+\cup\{0\}}; 1 \le i \le 8\Big\} \subseteq V$$

and

$$W_5 = \Big\{\Big(0 \,|\, L_{a_1} \;\; 0 \,|\, L_{a_2} \;\; 0 \;\; L_{a_3} \,|\, L_{a_4} \;\; 0 \;\; L_{a_5} \;\; 0 \,|\, L_{a_6} \;\; L_{a_7} \,|\, L_{a_8} \;\; 0\Big)\Big|$$

$$L_{a_i} \in L_{R^+\cup\{0\}}; 1 \le i \le 8\Big\} \subseteq V$$

be super semivector subspaces of V of refined labels over $L_{R^+\cup\{0\}}$. We see V = $\bigcup_{i=1}^{5} W_i$ but $W_i \cap W_j \neq$ {(0 | 0 0 | 0 0 0 | 0 0 0 0 | 0 0 | 0 0)} if i ≠ j, 1 ≤ i, j ≤ 5. Thus we see V is only a pseudo direct sum of super semivector subspaces of V over L_R and is not a direct sum of super semivector subspaces of V over L_R.

Example 5.15: Let

$$V = \Big\{\Big(L_{a_1} \,|\, L_{a_2} \;\; L_{a_3} \,|\, L_{a_4} \,|\, L_{a_5} \,|\, L_{a_6} \;\; L_{a_7} \,|\, L_{a_8}\Big)\Big| L_{a_i} \in L_{Q^+\cup\{0\}}; 1 \le i \le 8\Big\}$$

be a super semivector space of refined labels over the refined labels semifield $L_{Q^+\cup\{0\}}$.

Take

$$P_1 = \Big\{\Big(0 \,|\, 0 \;\; 0 \,|\, L_{a_1} \,|\, L_{a_2} \,|\, 0 \;\; 0 \,|\, L_{a_3}\Big)\Big| L_{a_i} \in L_{Q^+\cup\{0\}}; 1 \le i \le 3\Big\} \subseteq V,$$

$$P_2 = \Big\{\Big(L_{a_1} \,|\, 0 \;\; L_{a_2} \,|\, L_{a_3} \,|\, 0 \,|\, L_{a_4} \;\; 0 \,|\, 0\Big)\Big| L_{a_i} \in L_{Q^+\cup\{0\}}; 1 \le i \le 4\Big\} \subseteq V,$$

$$P_3 = \Big\{\Big(L_{a_1} \,|\, 0 \;\; L_{a_2} \,|\, L_{a_5} \,|\, 0 \,|\, L_{a_3} \;\; L_{a_4} \,|\, 0\Big)\Big| L_{a_i} \in L_{Q^+\cup\{0\}}; 1 \le i \le 5\Big\} \subseteq V,$$

$$P_4 = \Big\{\Big(0 \,|\, L_{a_1} \;\; L_{a_2} \,|\, L_{a_5} \,|\, L_{a_3} \,|\, 0 \;\; L_{a_4} \,|\, L_{a_6}\Big)\Big| L_{a_i} \in L_{Q^+\cup\{0\}}; 1 \le i \le 6\Big\}$$

and

$P_5 = \left\{\left(0 \middle| L_{a_1} \;\; 0 \middle| L_{a_2} \middle| 0 \middle| L_{a_3} \;\; 0 \middle| L_{a_4}\right) \middle| L_{a_i} \in L_{Q^+\cup\{0\}}; 1 \leq i \leq 4\right\} \subseteq V$

be super semivector subspaces of V of refined labels over $L_{Q^+\cup\{0\}}$. We see $V = \bigcup_{i=1}^{5} P_i$ and $P_i \cap P_j \neq \{(0 \mid 0\ 0 \mid 0 \mid 0 \mid 0 \mid 0\ 0 \mid 0)\}$, if $i \neq j$, $1 \leq i, j \leq 8$. Thus V is only a pseudo direct sum of super semivector subspaces $P_1, P_2, \ldots, P_5$ of V over $L_{Q^+\cup\{0\}}$.

Now it may so happen sometimes for any given super semivector space V of refined labels over the semifield $L_{Q^+\cup\{0\}}$ or $L_{R^+\cup\{0\}}$ the given set of super semi vector subspaces of V may not be say as $V \neq \bigcup_{i=1}^{n} P_i$ only $\bigcup_{i=1}^{n} P_i \not\subseteq V$. This situation can occur when we are supplied with the semivector subspaces of refined labels. But if we consider the super semivector subspace of refined labels we see that we can have either the direct union or the pseudo direct union. Now we discuss about the basis. Since we do not have negative elements in a semifield we need to apply some modifications in this regard.

Let $V = \left\{\left(L_{a_1} \;\; L_{a_2} \middle| L_{a_3} \;\; L_{a_4} \middle| \ldots \middle| L_{a_n}\right) \middle| L_{a_i} \in L_{R^+\cup\{0\}}; 1 \leq i \leq n\right\}$ be a super vector semivector space of refined labels over $L_{R^+\cup\{0\}}$. We see $B = \{[L_{m+1}\ 0 \mid 0\ 0 \mid \ldots \mid 0], [0\ L_{m+1} \mid 0\ 0 \mid \ldots \mid 0], \ldots, [0\ 0 \mid 0\ 0 \mid \ldots \mid L_{m+1}]\}$ acts as a basis of V over $L_{R^+\cup\{0\}}$. We define the notion of linearly dependent and independent in a different way [42-45]

For if $(v_1, \ldots, v_m)$ are super semivectors from the super semivector space V over $L_{R^+\cup\{0\}}$. We declare v_i's are linearly dependent if $v_i = \sum L_{a_j} v_j$; $L_{a_j} \in L_{R^+\cup\{0\}}$, $1 \leq j \leq m, j \neq i$, other wise linearly independent.

For take $V = \left\{\left(L_{a_1} \;\; L_{a_2} \middle| L_{a_3} \;\; L_{a_4} \middle| L_{a_5}\right) \middle| L_{a_i} \in L_{R^+\cup\{0\}}; 1 \leq i \leq 5\right\}$ to be a super semivector space of refined labels over $L_{R^+\cup\{0\}}$. Consider $(L_{m+1}\ 0 \mid 0\ 0 \mid L_{m+1})$ and $(L_{a_n}\ \ 0 \mid 0\ 0 \mid L_{a_n})$ in V.

Clearly $(L_{a_n}\ 0 \mid 0\ 0 \mid L_{a_n}) = L_{a_n}\ (L_{m+1}\ 0 \mid 0\ 0 \mid L_{m+1})$ so, the given two super semivectors in V are linearly dependent over $L_{R^+ \cup \{0\}}$. Now consider $(L_{m+1}\ 0 \mid 0\ 0 \mid 0)$, $(0\ 0 \mid L_{a_1}\ 0 \mid 0)$ and $(0\ 0 \mid 0\ 0 \mid L_{b_1})$ in V we see this set of super semivectors in V are linearly independent in V over $L_{R^+ \cup \{0\}}$.

Thus with this simple concept of linearly independent super semivectors in V we can define a basis of V as a set of linearly independent super semivectors in V which can generate the super semivector space V over $L_{R^+ \cup \{0\}}$.

Consider

$$V = \left\{ \left(L_{a_1} \mid L_{a_2} \mid L_{a_3}\ \ L_{a_4} \mid L_{a_5}\ \ L_{a_6} \right) \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 6 \right\}$$

to be a super semivector space of refined labels over $L_{R^+ \cup \{0\}}$. Consider the set B = $\{(L_{m+1} \mid 0 \mid 0\ 0 \mid 0\ 0), (0 \mid L_{m+1} \mid 0\ 0 \mid 0\ 0), (0 \mid 0 \mid L_{m+1}\ 0 \mid 0\ 0), (0 \mid 0 \mid 0\ L_{m+1} \mid 0\ 0), (0 \mid 0 \mid 0\ 0 \mid L_{m+1}\ 0), (0 \mid 0 \mid 0\ 0 \mid 0\ L_{m+1})\} \subseteq V$ is a set of linearly independent elements of V which generate V. Infact B is a basis of the super semi vector space V over $L_{R^+ \cup \{0\}}$. We can give more examples.

We would give the definition of super column vector space of refined labels over $L_{R^+ \cup \{0\}}$.

DEFINITION 5.2: *Let*

$$V = \left\{ \left. \begin{bmatrix} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ \hline \cdots \\ \hline L_{a_{n-1}} \\ L_{a_n} \end{bmatrix} \right| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq n \right\}$$

be an additive semigroup of super column vectors of refined labels. Clearly V is a super column semivector space of refined labels over L_R.

We will illustrate this situation by some examples.

Example 5.16: Let

$$V = \left\{ \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ L_{a_6} \end{array} \right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 6 \right\}$$

be a super column semivector space of refined labels over $L_{R^+ \cup \{0\}}$.

Example 5.17: Let

$$V = \left\{ \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ \hline L_{a_8} \\ L_{a_9} \\ L_{a_{10}} \\ \hline L_{a_{11}} \\ L_{a_{12}} \\ \hline L_{a_{13}} \end{array} \right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 13 \right\}$$

be a super column semivector space of refined labels over $L_{R^+ \cup \{0\}}$.

Example 5.18: Let

$$P = \left\{ \left. \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ L_{a_4} \\ \hline L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ \hline L_{a_8} \\ L_{a_9} \\ \hline L_{a_{10}} \end{array} \right] \right| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \le i \le 10 \right\}$$

be the class of super column semivector space of refined labels over $L_{Q^+ \cup \{0\}}$. It is to be noted P is not a super column semivector space of refined labels over $L_{R^+ \cup \{0\}}$. As in case of general semivector spaces the definition depends on the semifield over which the semivector space is defined.

We will illustrate this situation also by some examples.

Example 5.19: Let

$$V = \left\{ \left. \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \\ \hline L_{a_9} \\ L_{a_{10}} \end{array} \right] \right| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 10 \right\}$$

be a super column semivector space of refined labels over the semifield $L_{Q^+ \cup \{0\}}$.

Example 5.20: Let

$$P = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ \hline L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ \hline L_{a_6} \\ L_{a_7} \\ \hline L_{a_8} \\ L_{a_9} \end{array} \right] \middle| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \le i \le 9 \right\}$$

is a super column semivector space of refined labels over $L_{Q^+ \cup \{0\}}$.

Now we will give examples of basis and dimension of these spaces.

Example 5.21: Let

$$V = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \end{array} \right] \middle| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \le i \le 8 \right\}$$

be a super column vector space of refined labels over $L_{Q^+ \cup \{0\}}$.

Consider

$$M = \left\{ \left[\begin{array}{c} L_{m+1} \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ \hline L_{m+1} \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ \hline 0 \\ L_{m+1} \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right] \left[\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ \hline L_{m+1} \\ 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right] \left[\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ L_{m+1} \\ 0 \\ \hline 0 \\ 0 \end{array}\right] \left[\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ L_{m+1} \\ \hline 0 \\ 0 \end{array}\right] \left[\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline L_{m+1} \\ 0 \end{array}\right] \left[\begin{array}{c} 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ L_{m+1} \end{array}\right] \right\}$$

$\subseteq$ V is a basis of V over $L_{Q^+ \cup \{0\}}$.

Example 5.22: Let

$$V = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ \hline L_{a_8} \\ L_{a_9} \end{array}\right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 9 \right\}$$

be a super column semivector space of refined labels over $L_{R^+ \cup \{0\}}$. Clearly V is of dimension nine over $L_{R^+ \cup \{0\}}$.

Consider V as a super column semivector space of refined labels over $L_{Q^+ \cup \{0\}}$, then we see V is of infinite dimension over $L_{Q^+ \cup \{0\}}$. This clearly shows that as we change the semifield over which the super column semivector space is defined is

changed then we see the dimension of the space varies as the semifield over which V is defined varies.

Example 5.23: Let

$$V = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ \hline L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ \hline L_{a_6} \end{array} \right] \middle| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \le i \le 6 \right\}$$

be a super semivector space of refined labels over $L_{Q^+ \cup \{0\}}$ of dimension six over $L_{Q^+ \cup \{0\}}$. Clearly V is not defined over the refined label field $L_{R^+ \cup \{0\}}$.

Example 5.24: Let

$$M = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ \hline L_{a_3} \\ L_{a_4} \\ L_{a_5} \end{array} \right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 5 \right\}$$

be a super column semivector space of refined labels over the field $L_{R^+ \cup \{0\}}$.

Clearly M is of dimension five over $L_{R^+ \cup \{0\}}$; but M is of dimension infinite over $L_{Q^+ \cup \{0\}}$.

Now we proceed onto give examples of linear transformations on super column semivector space of refined labels.

Example 5.25: Let

$$V = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ \hline L_{a_6} \\ L_{a_7} \end{array}\right] \middle|\, L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 7 \right\}$$

and

$$W = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \\ L_{a_9} \\ L_{a_{10}} \\ \hline L_{a_{11}} \\ L_{a_{12}} \\ \hline L_{a_{13}} \end{array}\right] \middle|\, L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 13 \right\}$$

be two super column semivector spaces defined over the same semifield $L_{Q^+ \cup \{0\}}$.

Define a map $\eta : V \to W$ given by

$$\eta\left(\begin{bmatrix} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ \hline L_{a_6} \\ L_{a_7} \end{bmatrix}\right) = \begin{bmatrix} L_{a_1} \\ \hline L_{a_2} \\ L_{a_2} \\ \hline L_{a_3} \\ L_{a_3} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ \hline L_{a_7} \\ L_{a_7} \\ \hline L_{a_1} \end{bmatrix}.$$

It is easily verified η is a linear transformation of super semivector spaces or super semivector space linear transformation of V to W.

Example 5.26: Let

$$V = \left\{\left[L_{a_1} \;\; L_{a_2} \;\middle|\; L_{a_3} \;\middle|\; L_{a_4} \;\; L_{a_5} \;\middle|\; L_{a_6} \;\; L_{a_7} \;\; L_{a_8}\right] \middle| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \leq i \leq 7\right\}$$

and

$$W = \left\{\begin{bmatrix} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \\ L_{a_9} \end{bmatrix} \middle| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \leq i \leq 9\right\}$$

be two super semivector spaces of refined labels over $L_{Q^+ \cup \{0\}}$. Let η: V $\to$ W be defined by

$$\eta\left(\left(L_{a_1} \; L_{a_2} \left| L_{a_3} \right| L_{a_4} \; L_{a_5} \left| L_{a_6} \; L_{a_7} \; L_{a_8}\right.\right)\right) = \begin{bmatrix} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \\ 0 \end{bmatrix}.$$

It is easily verified η is a super semivector space linear transformation of V into W.

Example 5.27: Let

$$P = \left\{ \begin{bmatrix} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ \hline L_{a_8} \\ L_{a_9} \\ L_{a_{10}} \end{bmatrix} \middle| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \le i \le 10 \right\}$$

be a super semivector space of refined labels over $L_{Q^+ \cup \{0\}}$. Consider T : V $\to$ V defined by

$$
T\left(\left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ \hline L_{a_8} \\ L_{a_9} \\ L_{a_{10}} \end{array}\right]\right) = \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ 0 \\ \hline L_{a_4} \\ 0 \\ L_{a_5} \\ \hline L_{a_3} \\ \hline 0 \\ L_{a_6} \\ 0 \end{array}\right];
$$

T is a linear operator of super semivector spaces. Now having seen examples of linear transformations and linear operators on super column semi vector spaces of refined labels. We proceed onto define super matrix semivector spaces.

DEFINITION 5.4: *Let*

$$
V = \left\{ \left[\begin{array}{c|c|c} L_{a_{11}} & L_{a_{12}} & \ldots \quad L_{a_{1m}} \\ L_{a_{21}} & L_{a_{22}} & \ldots \quad L_{a_{2m}} \\ \hline \vdots & \vdots & \qquad\quad \vdots \\ \hline L_{a_{n1}} & L_{a_{n2}} & \ldots \quad L_{a_{nm}} \end{array}\right] \middle| \; L_{a_i} \in L_{R^+\cup\{0\}} (\text{ or } L_{Q^+\cup\{0\}});\right.
$$

$1 \leq i \leq n$ and $1 \leq j \leq m$} be the additive semigroup of super matrices of refined labels. V is a super matrix semivector space of refined labels over $L_{R^+\cup\{0\}}$.

Example 5.28: Let

$$
M = \left\{ \left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \end{array}\right] \middle| \; L_{a_i} \in L_{R^+\cup\{0\}}; 1 \leq i \leq 18 \right\}
$$

be a super matrix semivector space of refined labels over $L_{R^+\cup\{0\}}$. M is also known as the super column vector semivector space of refined labels over the field of refined labels $L_{R^+\cup\{0\}}$.

Example 5.29: Let

$$V = \left\{ \left[\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \end{array} \right] \middle| L_{a_i} \in L_{R^+\cup\{0\}}; 1 \le i \le 9 \right\}$$

be the super square matrix semivector space of refined labels over $L_{R^+\cup\{0\}}$.

Example 5.30: Let

$$W = \left\{ \left[\begin{array}{c|cc|ccc|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ L_{a_{19}} & L_{a_{20}} & L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} & L_{a_{25}} & L_{a_{26}} & L_{a_{27}} \\ L_{a_{28}} & L_{a_{29}} & L_{a_{30}} & L_{a_{31}} & L_{a_{32}} & L_{a_{33}} & L_{a_{34}} & L_{a_{35}} & L_{a_{36}} \end{array} \right] \right|$$

$\left. L_{a_i} \in L_{R^+\cup\{0\}}; 1 \le i \le 36 \right\}$ be a super row vector semivector space of refined labels over $L_{R^+\cup\{0\}}$ or super matrix semivector space of refined labels over $L_{R^+\cup\{0\}}$.

Example 5.31: Let

$$V = \left\{ \left[\begin{array}{c|cc|ccc|c|c} L_{a} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \\ \hline L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} & L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ L_{a_{25}} & L_{a_{26}} & L_{a_{27}} & L_{a_{28}} & L_{a_{29}} & L_{a_{30}} & L_{a_{31}} & L_{a_{32}} \\ L_{a_{33}} & L_{a_{34}} & L_{a_{35}} & L_{a_{36}} & L_{a_{37}} & L_{a_{38}} & L_{a_{39}} & L_{a_{40}} \\ \hline L_{a_{41}} & L_{a_{42}} & L_{a_{43}} & L_{a_{44}} & L_{a_{45}} & L_{a_{46}} & L_{a_{47}} & L_{a_{48}} \end{array} \right] \right|$$

$L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 48\}$ be a super matrix vector space of refined labels over the semifield of refined labels $L_{R^+ \cup \{0\}}$.

We as in case of other super semivector spaces define the notion of super semivector subspaces, basis, dimension and linear transformations. However this task is direct and hence is left as an exercise to the reader we only give examples of them.

Example 5.32: Let

$$V = \left\{ \left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \end{array} \right] \middle| \, L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 18 \right\}$$

be a super matrix semivector space or super column vector semivector space over $L_{R^+ \cup \{0\}}$. Clearly

$$P = \left\{ \left[\begin{array}{ccc} 0 & 0 & 0 \\ \hline L_{a_1} & L_{a_2} & L_{a_3} \\ 0 & 0 & 0 \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ 0 & 0 & 0 \\ L_{a_7} & L_{a_8} & L_{a_9} \end{array} \right] \middle| \, L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 9 \right\} \subseteq V$$

is a super column vector, semivector subspace of refined labels of V over $L_{R^+ \cup \{0\}}$. We have several but only finite number of super matrix semivector subspaces of V over $L_{R^+ \cup \{0\}}$. If V be defined over the refined semifield of labels $L_{Q^+ \cup \{0\}}$ then V has infinite number of super matrix semivector subspaces. Thus even the number of super matrix semivector subspaces depends on the semifield over which it is defined. This is evident from example 5.32.

Example 5.33: Let

$$V = \left\{ \left[\begin{array}{c|cc|cc|cc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} \\ L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} \\ \hline L_{a_{15}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \\ \hline L_{a_{22}} & L_{a_{23}} & L_{a_{24}} & L_{a_{25}} & L_{a_{26}} & L_{a_{27}} & L_{a_{28}} \\ L_{a_{29}} & L_{a_{30}} & L_{a_{31}} & L_{a_{32}} & L_{a_{33}} & L_{a_{34}} & L_{a_{35}} \\ \hline L_{a_{36}} & L_{a_{37}} & L_{a_{38}} & L_{a_{39}} & L_{a_{40}} & L_{a_{41}} & L_{a_{42}} \end{array} \right] \middle| \begin{array}{c} L_{a_i} \in L_{Q^+ \cup \{0\}}; \\ 1 \leq i \leq 42 \end{array} \right\}$$

be a super matrix of semi vector space of refined labels over $L_{Q^+ \cup \{0\}}$. V is a finite dimensional super matrix semivector subspace over $L_{Q^+ \cup \{0\}}$ having infinite number of semivector subspaces of refined labels over $L_{Q^+ \cup \{0\}}$.

Now we proceed onto define new class of super matrix semivector spaces over the integer semifield $Z^+ \cup \{0\}$; known as the integer super matrix semivector space of refined labels.

We will illustrate them before we proceed onto derive properties related with them.

Example 5.34: Let

$$V = \left\{ \left(L_{a_1} \mid L_{a_2} \mid L_{a_3} \;\; L_{a_4} \;\; L_{a_5} \mid L_{a_6} \;\; L_{a_7} \right) \middle| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \leq i \leq 7 \right\}$$

be a integer super row semivector space of refined labels over $Z^+ \cup \{0\}$, the semifield of integers.

Example 5.35: Let

$$V = \left\{ \left[\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \end{array} \right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 15 \right\}$$

be an integer super matrix semivector space of refined labels over $Z^+ \cup \{0\}$, the semifield of integers.

Example 5.36: Let

$$V = \left\{ \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ L_{a_4} \\ L_{a_5} \\ \hline L_{a_6} \\ L_{a_7} \\ \hline L_{a_8} \end{array}\right] \middle| \; L_{a_i} \in L_{R^+ \cup \{0\}} \right\}$$

be an integer super row semivector space over the semifield $Z^+ \cup \{0\}$.

Example 5.37: Let

$$V = \left\{ \left[\begin{array}{c|ccc|cccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \\ L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} & L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ L_{a_{25}} & L_{a_{26}} & L_{a_{27}} & L_{a_{28}} & L_{a_{29}} & L_{a_{30}} & L_{a_{31}} & L_{a_{32}} \\ L_{a_{33}} & L_{a_{34}} & L_{a_{35}} & L_{a_{36}} & L_{a_{37}} & L_{a_{38}} & L_{a_{39}} & L_{a_{40}} \end{array}\right] \middle| \right.$$

$\left. L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \leq i \leq 40 \right\}$ be a integer super matrix semivector space of refined labels over the semifield $Z^+ \cup \{0\}$.

We can define substructures, linear transformation a basis; this task can be done as a matter of routine by the interested reader.

But we illustrate all these situations by some examples.

Example 5.38: Let

$$M = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ \hline L_{a_8} \\ L_{a_9} \\ L_{a_{10}} \end{array} \right] \middle| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \le i \le 10 \right\}$$

be an integer super column semivector space of refined labels over the semifield $Z^+ \cup \{0\}$.

Take

$$K = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline 0 \\ L_{a_2} \\ \hline 0 \\ L_{a_3} \\ 0 \\ \hline L_{a_4} \\ \hline 0 \\ L_{a_5} \\ 0 \end{array} \right] \middle| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \le i \le 5 \right\} \subseteq M,$$

K is a integer super column semivector subspaces of refined labels over the semifield $Z^+ \cup \{0\}$.

Infact we can have infinite number of integer super column semivector subspaces of V of refined labels over $Z^+ \cup \{0\}$.

For take

$$P = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ \hline L_{a_8} \\ L_{a_9} \\ L_{a_{10}} \end{array} \right] \middle| L_{a_i} \in L_T \text{ where } 6Z^+ \cup \{0\}; 1 \leq i \leq 10 \right\} \subseteq M$$

is again an integer super column semivector subspace of M of refined labels over $Z^+ \cup \{0\}$. Infact 6 can be replaced by any positive integer in Z^+ and K will continue to be a super column semivector subspace of M of refined labels over $Z^+ \cup \{0\}$.

Example 5.39: Let

$$P = \left\{ \left(L_{a_1} \;\; L_{a_2} \;\; L_{a_3} \,\middle|\, L_{a_4} \;\; L_{a_5} \,\middle|\, L_{a_6} \right) \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 6 \right\}$$

be an integer super row semivector space of refined label over the semifield $Z^+ \cup \{0\}$.

$$K = \left\{ \left(L_{a_1} \;\; L_{a_2} \;\; L_{a_3} \,\middle|\, 0 \;\; 0 \,\middle|\, L_{a_4} \right) \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 4 \right\} \subseteq P \text{ is}$$

an integer super row semivector subspace of P of refined labels over the integer semifield $Z^+ \cup \{0\}$.

Example 5.40: Let

$$S = \left\{ \left[\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \end{array} \right] \middle| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \leq i \leq 15 \right\}$$

be an integer super matrix semivector space of refined labels over the semifield $Z^+ \cup \{0\}$.

Consider

$$H = \left\{ \left[\begin{array}{cc|c} 0 & 0 & L_{a_1} \\ 0 & 0 & L_{a_2} \\ \hline L_{a_3} & L_{a_4} & 0 \\ L_{a_5} & L_{a_6} & 0 \\ L_{a_7} & L_{a_8} & 0 \end{array} \right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 8 \right\} \subseteq S;$$

H is an integer super matrix semivector subspace of S over the integer semifield $Z^+ \cup \{0\}$.

It is pertinent to mention here that S has infinitely many integer super matrix semivector subspace of refined labels over the integer semifield $Z^+ \cup \{0\}$.

Example 5.41: Let

$$V = \left\{ \left(L_{a_1} \mid L_{a_2} \;\; L_{a_3} \;\; L_{a_4} \mid L_{a_5} \;\; L_{a_6} \right) \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 6 \right\}$$

be an integer super row vector semivector space of refined labels over the semifield of integers $Z^+ \cup \{0\}$.

Take

$$W_1 = \left\{ \left(L_{a_1} \mid L_{a_2} \;\; 0 \;\; 0 \mid 0 \;\; 0 \right) \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 2 \right\} \subseteq V,$$

$$W_2 = \left\{ \left(0 \mid 0 \;\; L_{a_1} \;\; 0 \mid 0 \;\; 0 \right) \middle| L_{a_i} \in L_{R^+ \cup \{0\}} \right\} \subseteq V,$$

$$W_3 = \left\{ \left(0 \mid 0 \;\; 0 \;\; L_{a_1} \mid L_{a_2} \;\; 0 \right) \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 2 \right\} \subseteq V$$

and

$$W_4 = \left\{ \left(0 \mid 0 \;\; 0 \;\; 0 \mid 0 \;\; L_{a_1} \right) \middle| L_{a_i} \in L_{R^+ \cup \{0\}} \right\} \subseteq V$$

be integer super row vector semi subspaces of V.

Clearly $V = \bigcup_{i=1}^{4} W_i$; with $W_i \cap W_j = (0 \mid 0\ 0\ 0 \mid 0\ 0)$; $i \neq j$, $1 \le i, j \le 4$. V is the direct union of integer super row semivector subspaces of V over the semifield $Z^+ \cup \{0\}$.

Example 5.42: Let

$$V = \left\{ \left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ \hline L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \\ L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \end{array}\right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 24 \right\}$$

be an integer super column vector semivector space of refined labels over the semifield $Z^+ \cup \{0\} = S$. Consider

$$V_1 = \left\{ \left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 3 \right\} \subseteq V,$$

$$V_2 = \left\{ \left[\begin{array}{ccc} 0 & 0 & 0 \\ \hline L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 6 \right\} \subseteq V,$$

$$V_3 = \left\{ \left[\begin{array}{ccc} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 9 \right\} \subseteq V$$

and

$$V_4 = \left\{ \left[\begin{array}{ccc} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \end{array} \right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 6 \right\} \subseteq V$$

be integer super column vector semivector subspaces of V over the semifield $S = Z^+ \cup \{0\}$. Clearly $V = \bigcup_{i=1}^{4} V_i$ with

$$V_i \cap V_j = \left[\begin{array}{ccc} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \text{ if } i \neq j,\ 1 \leq i, j \leq 4.$$

Thus V is a direct sum of integer super column vector semivector subspaces of V over S.

Example 5.43: Let

$$V = \left\{ \left[\begin{array}{c|cc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \end{array} \right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 18 \right\}$$

be an integer super matrix semivector space of refined labels over the semifield S = $Z^+ \cup \{0\}$.

Consider

$$P_1 = \left\{ \left[\begin{array}{c|cc} L_{a_1} & L_{a_2} & 0 \\ \hline 0 & L_{a_3} & 0 \\ L_{a_4} & 0 & 0 \\ \hline 0 & 0 & L_{a_6} \\ 0 & 0 & 0 \\ 0 & L_{a_5} & 0 \end{array} \right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 6 \right\} \subseteq V,$$

$$P_2 = \left\{ \left[\begin{array}{c|cc} L_{a_1} & 0 & L_{a_2} \\ \hline 0 & 0 & 0 \\ L_{a_3} & L_{a_4} & 0 \\ \hline 0 & 0 & L_{a_6} \\ 0 & 0 & 0 \\ L_{a_5} & 0 & 0 \end{array} \right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 6 \right\} \subseteq V,$$

$$P_3 = \left\{ \left[\begin{array}{c|cc} L_{a_1} & 0 & 0 \\ \hline L_{a_4} & L_{a_2} & L_{a_3} \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ L_{a_5} & L_{a_6} & 0 \end{array} \right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 6 \right\} \subseteq V,$$

$$P_4 = \left\{ \left[\begin{array}{c|cc} L_{a_1} & 0 & 0 \\ \hline 0 & 0 & 0 \\ L_{a_2} & 0 & L_{a_4} \\ \hline L_{a_5} & 0 & 0 \\ 0 & L_{a_7} & 0 \\ L_{a_6} & 0 & 0 \end{array} \right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 7 \right\} \subseteq V,$$

$$P_5 = \left\{ \left[\begin{array}{c|cc} L_{a_1} & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ L_{a_2} & 0 & 0 \\ L_{a_3} & 0 & 0 \end{array} \right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 3 \right\} \subseteq V$$

and
$$P_6 = \left\{ \left[\begin{array}{c|cc} L_{a_1} & L_{a_2} & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & L_{a_3} & 0 \\ 0 & 0 & L_{a_4} \\ 0 & 0 & L_{a_5} \end{array} \right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 5 \right\} \subseteq V$$

are integer super matrix semivector subspaces of refined labels of V over the semifield $Z^+ \cup \{0\}$.

Clearly $V = \bigcup_{i=1}^{6} P_i$ and $V_i \cap V_j \neq \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ if $i \neq j$, $1 \leq i, j \leq 6$.

Thus V is a pseudo direct sum of integer super matrix semivector subspaces of V over $Z^+ \cup \{0\}$ of refined labels.

Example 5.44: Let

$$B = \left\{ \left[\begin{array}{c|ccc|c} L_{a_1} & L_{a_6} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} \\ L_{a_2} & L_{a_7} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \\ L_{a_3} & L_{a_8} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} \\ L_{a_4} & L_{a_9} & L_{a_{20}} & L_{a_{21}} & L_{a_{22}} \\ L_{a_5} & L_{a_{10}} & L_{a_{23}} & L_{a_{24}} & L_{a_{25}} \end{array}\right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 25 \right\}$$

be a integer super row vector semivector space of refined labels over the semi field $Z^+ \cup \{0\}$.

Consider

$$H_1 = \left\{ \left[\begin{array}{c|ccc|c} L_{a_1} & 0 & 0 & 0 & 0 \\ L_{a_2} & 0 & 0 & 0 & 0 \\ L_{a_3} & 0 & 0 & 0 & L_{a_6} \\ L_{a_4} & 0 & 0 & 0 & 0 \\ L_{a_5} & 0 & 0 & 0 & 0 \end{array}\right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 6 \right\} \subseteq B,$$

$$H_2 = \left\{ \left[\begin{array}{c|ccc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & 0 \\ 0 & 0 & 0 & 0 & L_{a_5} \\ 0 & 0 & 0 & 0 & L_{a_6} \\ 0 & 0 & 0 & 0 & L_{a_7} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 7 \right\} \subseteq B,$$

$$H_3 = \left\{ \left[\begin{array}{c|ccc|c} 0 & L_{a_2} & 0 & 0 & 0 \\ L_{a_1} & L_{a_4} & L_{a_3} & L_{a_5} & L_{a_6} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 6 \right\} \subseteq B,$$

$$H_4 = \left\{ \left[\begin{array}{c|ccc|c} L_{a_1} & 0 & 0 & 0 & L_{a_9} \\ L_{a_2} & 0 & 0 & 0 & L_{a_{10}} \\ 0 & L_{a_3} & L_{a_4} & L_{a_5} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & L_{a_6} & L_{a_7} & L_{a_8} & 0 \end{array}\right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 10 \right\} \subseteq B,$$

and

$$H_5 = \left\{ \left[\begin{array}{c|ccc|c} 0 & L_{a_2} & 0 & 0 & 0 \\ L_{a_1} & 0 & L_{a_3} & 0 & 0 \\ 0 & 0 & 0 & L_{a_4} & 0 \\ 0 & L_{a_5} & L_{a_6} & L_{a_7} & 0 \\ 0 & 0 & 0 & L_{a_8} & L_{a_9} \end{array}\right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 9 \right\} \subseteq B,$$

be an integer super row vector semivector subspaces of B of refined labels over the integer semifield $S = Z^+ \cup \{0\}$.

$B = \bigcup_{i=1}^{5} H_i$ and

$$H_i \cap H_j \neq \left[\begin{array}{c|ccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ if } i \neq j,\ 1 \leq i, j \leq 5.$$

Thus H is only a pseudo direct sum of integer super row vector semivector subspaces of B of refined labels over $Z^+ \cup \{0\}$.

Now having seen examples of pseudo direct sum of integer super matrix semivector subspaces and direct sum of integer super matrix semivector subspaces we now proceed onto give examples of integer linear transformation and integer linear operator of integer super matrix semivector spaces defined over the integer semifield $Z^+ \cup \{0\}$.

Example 5.45: Let

$$V = \left\{ \left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ \hline L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \end{array} \right] \middle| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \le i \le 21 \right\}$$

be an integer super matrix semivector space of refined labels over the integer semifield $Z^+ \cup \{0\}$.

$$W = \left\{ \left[\begin{array}{c|cc|ccc|cc} L_{a_1} & L_{a_4} & L_{a_7} & L_{a_{10}} & L_{a_{13}} & L_{a_{16}} & L_{a_{19}} & L_{a_{22}} \\ L_{a_2} & L_{a_5} & L_{a_8} & L_{a_{11}} & L_{a_{14}} & L_{a_{17}} & L_{a_{20}} & L_{a_{23}} \\ L_{a_3} & L_{a_6} & L_{a_9} & L_{a_{12}} & L_{a_{15}} & L_{a_{18}} & L_{a_{21}} & L_{a_{24}} \end{array} \right] \middle| \begin{array}{c} L_{a_i} \in L_{R^+ \cup \{0\}}; \\ 1 \le i \le 24 \end{array} \right\}$$

be an integer super matrix semivector space of refined labels over the integer semifield $Z^+ \cup \{0\}$. Define T a integer linear transformation from V into W as follows.

$$T \left(\left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ \hline L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \end{array} \right] \right)$$

$$= \left[\begin{array}{c|cc|ccc|cc} L_{a_1} & L_{a_4} & L_{a_7} & L_{a_{10}} & L_{a_{13}} & L_{a_{16}} & L_{a_{19}} & 0 \\ L_{a_2} & L_{a_5} & L_{a_8} & L_{a_{11}} & L_{a_{14}} & L_{a_{17}} & L_{a_{20}} & 0 \\ L_{a_3} & L_{a_6} & L_{a_9} & L_{a_{12}} & L_{a_{15}} & L_{a_{18}} & L_{a_{21}} & 0 \end{array}\right].$$

Example 5.46: Let

$$V = \left\{ \left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ \hline L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \\ \hline L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ L_{a_{25}} & L_{a_{26}} & L_{a_{27}} \end{array}\right] \middle| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \leq i \leq 27 \right\}$$

be an integer super semivector space of refined labels over the semifield $S = Z^+ \cup \{0\}$.

Define $T : V \to V$ by

$$T\left(\left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ \hline L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \\ \hline L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ L_{a_{25}} & L_{a_{26}} & L_{a_{27}} \end{array}\right]\right) = \left[\begin{array}{ccc} 0 & 0 & 0 \\ \hline L_{a_1} & L_{a_2} & L_{a_3} \\ 0 & 0 & 0 \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ 0 & 0 & 0 \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline 0 & 0 & 0 \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ 0 & 0 & 0 \end{array}\right].$$

Clearly T is a integer linear operator on V where kernel T is a nontrivial semivector subspace of V.

Now we give an example of a projection the concept is direct and can define it as in case of semivector spaces.

Example 5.47: Let

$$V = \left\{ \left[\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \end{array}\right] \middle| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \le i \le 18 \right\}$$

be an integer super matrix semivector space of refined labels over the integer semifield $S = Z^+ \cup \{0\}$.

Consider

$$W = \left\{ \left[\begin{array}{cc|c} 0 & L_{a_1} & 0 \\ \hline L_{a_2} & L_{a_3} & L_{a_4} \\ 0 & 0 & L_{a_5} \\ \hline L_{a_6} & L_{a_7} & 0 \\ 0 & 0 & 0 \\ L_{a_8} & L_{a_9} & 0 \end{array}\right] \middle| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \le i \le 9 \right\} \subseteq V$$

be an integer super matrix semivector space of refined labels over the semifield $Z^+ \cup \{0\}$ of V.

Define T an integer linear operator from V into V given by

$$T\left(\left[\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \end{array}\right] \right) = \left[\begin{array}{cc|c} 0 & L_{a_1} & 0 \\ \hline L_{a_2} & L_{a_3} & L_{a_4} \\ 0 & 0 & L_{a_5} \\ \hline L_{a_6} & L_{a_7} & 0 \\ 0 & 0 & 0 \\ L_{a_8} & L_{a_9} & 0 \end{array}\right].$$

Clearly T is a linear operator with nontrival kernel.

Further T (W) $\subseteq$ W that is W is invariant under the integer linear operator T on V. Consider

$$T_1 \left(\left[\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \end{array}\right] \right) = \left[\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline 0 & 0 & 0 \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline 0 & 0 & 0 \\ L_{a_7} & L_{a_8} & L_{a_9} \\ 0 & 0 & 0 \end{array}\right].$$

We see T_1 is a integer linear operator with nontrivial kernel but clearly T_1 is not an integer linear operator which keeps W invariant; that is T_1 (W) $\not\subseteq$ W.

Thus all integer linear operators in general need not keep W invariant.

Now we proceed onto define various types of super matrix semivector spaces of refined labels.

DEFINITION 5.4: *Let V = {set of super matrices whose entries are labels from $L_{R^+ \cup \{0\}}$ or $L_{Q^+ \cup \{0\}}$} be a set of super matrices refined labels. If V is a such that for a set S subset of positive reals ($R^+ \cup \{0\}$ or $Z^+ \cup \{0\}$ or $Q^+ \cup \{0\}$) sv and vs are in V for every $v \in V$ and $s \in S$; then we define V to be a set super matrix semivector space of refined labels over the set S or super matrix semivector space of refined labels over the set S, of subset of reals.*

We will first illustrate this situation by some examples.

Example 5.48: Let

$$V = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ \hline L_{a_3} \\ L_{a_4} \end{array}\right], \left[\begin{array}{c|c} L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \end{array}\right], \left(L_{a_1} \mid L_{a_2} \;\; L_{a_3} \mid L_{a_4} \mid L_{a_5} \mid L_{a_6} \;\; L_{a_7} \mid L_{a_8}\right) \right|$$

$\left. L_{a_i} \in L_{Q^+\cup\{0\}}; 1 \leq i \leq 8 \right\}$ be a set of super matrices of refined labels.

Let S = {a | a ∈ $3Z^+ \cup 2Z^+ \cup \{0\}$}. V is a set super matrix semivector space of refined labels over the set S.

Example 5.49: Let

$$V = \left\{ \left[\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \end{array}\right], \left[\begin{array}{c|ccc|cc} L_{a_1} & L_{a_4} & L_{a_7} & L_{a_{10}} & L_{a_{13}} & L_{a_{16}} \\ L_{a_2} & L_{a_5} & L_{a_8} & L_{a_{11}} & L_{a_{14}} & L_{a_{17}} \\ L_{a_3} & L_{a_6} & L_{a_9} & L_{a_{12}} & L_{a_{15}} & L_{a_{18}} \end{array}\right], \right.$$

$$\left. \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \end{array}\right] \left(L_{a_1} \,|\, L_{a_2} \;\; L_{a_3} \,|\, L_{a_4} \,|\, L_{a_5} \,|\, L_{a_6} \;\; L_{a_7} \,|\, L_{a_8} \right) \,\middle|\, \begin{array}{c} L_{a_i} \in L_{Q^+\cup\{0\}}; \\ 1 \leq i \leq 18 \end{array} \right\}$$

be a super matrix set semivector space over the set S = {a | a ∈ $3Z^+ \cup 25Z^+ \cup \{0\}$} ⊆ $L_{R^+\cup\{0\}}$ of refined labels.

Clearly V is of infinite order we can define two substructure for V, a super matrix set semivector space of refined labels over the set S, S ⊆ $R^+ \cup \{0\}$.

DEFINITION 5.5: *Let V be a super matrix semivector space of refined labels with entries from $L_{R^+\cup\{0\}}$ over the set S (S ⊆ $R^+ \cup$ {0}). Suppose W ⊆ V, W a proper subset of V and if W itself is a set super matrix semivector space over the set S then we define W to be a set super matrix semivector subspace of V of refined labels over the set S.*

We will first illustrate this situation by some examples.

Example 5.50: Let

$$V = \left\{ \left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} \end{array} \right], \left(\begin{array}{ccc|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array} \right), \left[\begin{array}{c|c|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \end{array} \right] \right|$$

$L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \leq i \leq 15 \Big\}$ be a set super matrix semivector space of refined labels over the set S = {a ∈ $3Z^+ \cup 5Z^+ \cup \{0\}\} \subseteq Q^+ \cup \{0\}$. Consider

$$W = \left\{ \left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ 0 & 0 \\ \hline 0 & 0 \\ L_{a_5} & L_{a_6} \end{array} \right], \left[\begin{array}{c|c|c} L_{a_1} & 0 & L_{a_6} \\ L_{a_2} & 0 & L_{a_7} \\ \hline L_{a_3} & 0 & L_{a_8} \\ L_{a_4} & 0 & L_{a_9} \\ L_{a_5} & 0 & L_{a_{10}} \end{array} \right] \right| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \leq i \leq 10 \right\}$$

$\subseteq$ V, W is a set super matrix semivector subspace of refined labels over the set S $\subseteq Q^+ \cup \{0\}$.

Example 5.51: Let

$$V = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ \hline L_{a_6} \\ L_{a_7} \end{array} \right], \left[\begin{array}{cc|c|ccc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_{13}} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{14}} \\ L_{a_{15}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \\ \hline L_{a_{22}} & L_{a_{23}} & L_{a_{24}} & L_{a_{25}} & L_{a_{26}} & L_{a_{27}} & L_{a_{28}} \\ L_{a_{29}} & L_{a_{30}} & L_{a_{31}} & L_{a_{32}} & L_{a_{33}} & L_{a_{34}} & L_{a_{35}} \end{array} \right], \left[\begin{array}{c|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_7} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array} \right] \right|$$

$L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \leq i \leq 35 \Big\}$ be a set super matrix semivector space of refined labels over the set S = $3Z^+ \cup 5Z^+ \cup 8Z^+ \cup \{0\}\} \subseteq Q^+ \cup \{0\}$.

Consider

$$P = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline 0 \\ L_{a_2} \\ \hline 0 \\ L_{a_3} \\ \hline L_{a_4} \\ 0 \end{array}\right], \left[\begin{array}{c|cc|c} L_{a_1} & 0 & 0 & L_{a_2} \\ \hline 0 & L_{a_8} & 0 & L_{a_6} \\ 0 & 0 & L_{a_9} & L_{a_7} \\ \hline L_{a_3} & L_{a_4} & L_{a_5} & L_{a_{16}} \end{array}\right], \left[\begin{array}{cc|c|ccc|c} 0 & 0 & L_{a_5} & 0 & 0 & 0 & L_{a_{11}} \\ \hline L_{a_1} & 0 & 0 & L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{12}} \\ 0 & L_{a_2} & L_{a_6} & 0 & 0 & 0 & 0 \\ \hline 0 & L_{a_3} & 0 & 0 & 0 & L_{a_{13}} & 0 \\ L_{a_4} & 0 & L_{a_7} & L_{a_{15}} & 0 & 0 & L_{a_{14}} \end{array}\right] \right|$$

$L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \le i \le 15\Big\} \subseteq V$; P is a set super matrix semivector subspace of V over the set S of refined labels.

Now we proceed onto define the notion of subset super semivector subspace of refined labels.

DEFINITION 5.6: *Let V be a set super matrix semivector space of refined labels over the set S, ($S \subseteq R^+ \cup \{0\}$). Let $W \subseteq V$; and $P \subseteq S$ (W and P are proper subsets of V and S respectively). If W is a super matrix semivector space of refined labels over the set P then we define W to be a subset super matrix semivector subspace of refined labels over the subset P of the set S.*

We will illustrate this situation by some examples.

Example 5.52: Let

$$P = \left\{ \left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \end{array}\right], \left[\begin{array}{c|ccc|cc} L_{a_1} & L_{a_4} & L_{a_7} & L_{a_{10}} & L_{a_{13}} & L_{a_{16}} \\ L_{a_2} & L_{a_5} & L_{a_8} & L_{a_{11}} & L_{a_{14}} & L_{a_{17}} \\ L_{a_3} & L_{a_6} & L_{a_9} & L_{a_{12}} & L_{a_{15}} & L_{a_{18}} \end{array}\right], \right.$$

$$\left.\left\{\left[\begin{array}{c|c|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right] \right| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \leq i \leq 21 \right\}$$

be a set super matrix semivector space of refined labels over the set

$$S = \{a \in 3Z^+ \cup 8Z^+ \cup 7Z^+ \cup 13Z^+ \cup \{0\}\} \subseteq Q^+ \cup \{0\}.$$

Consider

$$K = \left\{\left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ 0 & 0 & 0 \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ \hline 0 & 0 & 0 \\ L_{a_7} & L_{a_8} & L_{a_9} \\ 0 & 0 & 0 \\ L_{a_{11}} & L_{a_{12}} & L_{a_{13}} \end{array}\right], \left[\begin{array}{c|ccc|cc} L_{a_1} & 0 & L_{a_3} & 0 & L_{a_7} & 0 \\ 0 & L_{a_4} & 0 & L_{a_5} & 0 & L_{a_8} \\ L_{a_2} & 0 & L_{a_6} & 0 & L_{a_9} & 0 \end{array}\right], \right.$$

$$\left.\left[\begin{array}{c|c|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right] \right| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \leq i \leq 12 \right\} \subseteq P$$

and

$$T = a \in 3Z^+ \cup 7Z^+ \cup \{0\}\} \subseteq S \subseteq Q^+ \cup \{0\}.$$

K is a subset super matrix semivector subspace of P over the subset T of S.

Example 5.53: Let

$$V = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \\ \hline L_{a_9} \end{array}\right], \left(L_{a_1} \,|\, L_{a_2} \;\; L_{a_3} \,|\, L_{a_4} \;\; L_{a_5} \;\; L_{a_6} \,|\, L_{a_7} \;\; L_{a_8} \;\; L_{a_9}\right), \right.$$

$$\left. \left[\begin{array}{cc|c|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array}\right], \left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} \end{array}\right] \,\middle|\, L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \leq i \leq 16 \right\}$$

be a set super matrix semivector space of refined labels over the set S = a $\in$ $5Z^+ \cup 3Z^+ \cup 7Z^+ \cup 8Z^+ \cup 13Z^+ \cup 11Z^+ \cup \{0\}\}$.
Consider

$$W = \left\{ \left(L_{a_1} \,|\, 0 \;\; 0 \,|\, L_{a_2} \;\; L_{a_3} \;\; L_{a_4} \,|\, 0 \;\; L_{a_5} \;\; 0\right), \left[\begin{array}{c} 0 \\ \hline L_{a_1} \\ 0 \\ \hline L_{a_2} \\ 0 \\ 0 \\ \hline L_{a_3} \\ L_{a_4} \\ \hline L_{a_5} \end{array}\right], \right.$$

$$\left.\left.\left[\begin{array}{cc} L_{a_1} & 0 \\ 0 & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ \hline L_{a_5} & 0 \\ 0 & L_{a_6} \\ L_{a_7} & 0 \end{array}\right], \left[\begin{array}{cc|c|c} L_{a_1} & 0 & L_{a_2} & 0 \\ \hline 0 & L_{a_3} & 0 & L_{a_4} \\ 0 & 0 & 0 & 0 \\ \hline L_{a_5} & 0 & L_{a_6} & 0 \end{array}\right] \right| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \leq i \leq 7 \right\} \subseteq V$$

and K = a ∈ $3Z^+ \cup 8Z^+ \cup 13Z^+ \cup \{0\}\} \subseteq S \subseteq Q^+ \cup \{0\}$. Clearly W is a subset super matrix semivector subspace of V of refined labels over the subset K of S.

As in case of usual vector spaces we can define direct union and pseudo direct union of set super matrix semivector space. The definition is easy and direct and hence is left for the reader as an exercise.

We now illustrate these two situations by some examples.

Example 5.54: Let

$$V = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ \hline L_{a_8} \\ L_{a_9} \\ \hline L_{a_{10}} \end{array}\right], \left[\begin{array}{c|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \\ L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \\ \hline L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \end{array}\right], \left[\begin{array}{c|ccc|cc} L_{a_1} & L_{a_4} & L_{a_7} & L_{a_{10}} & L_{a_{13}} & L_{a_{16}} \\ L_{a_2} & L_{a_5} & L_{a_8} & L_{a_{11}} & L_{a_{14}} & L_{a_{17}} \\ L_{a_3} & L_{a_6} & L_{a_9} & L_{a_{12}} & L_{a_{15}} & L_{a_{18}} \end{array}\right] \right|$$

$L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \leq i \leq 24 \}$ be a set super matrix of semivector space of refined labels over the set S = {a ∈ $3Z^+ \cup 25Z^+ \cup 11Z^+ \cup 32Z^+ \cup \{0\}\} \subseteq Q^+ \cup \{0\}$.

Consider

$$W_1 = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ \hline L_{a_8} \\ L_{a_9} \\ \hline L_{a_{10}} \end{array} \right] \middle| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \leq i \leq 10 \right\} \subseteq V,$$

$$W_2 = \left\{ \left[\begin{array}{c|ccc|cc} L_{a_1} & L_{a_4} & L_{a_7} & L_{a_{10}} & L_{a_{13}} & L_{a_{16}} \\ L_{a_2} & L_{a_5} & L_{a_8} & L_{a_{11}} & L_{a_{14}} & L_{a_{17}} \\ L_{a_3} & L_{a_6} & L_{a_9} & L_{a_{12}} & L_{a_{15}} & L_{a_{18}} \end{array} \right] \middle| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \leq i \leq 18 \right\}$$

$\subseteq$ V and

$$W_3 = \left\{ \left[\begin{array}{c|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \\ L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \\ \hline L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \end{array} \right] \middle| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \leq i \leq 24 \right\} \subseteq V$$

be set super matrix semivector subspaces of refined labels over the set S.

Now $V = \bigcup_{i=1}^{3} W_i$ and $W_i \cap W_j = \phi$ if $i \neq j$, $1 \leq i, j \leq 3$. Thus V is the direct union of set super matrix semivector subspaces.

Example 5.55: Let

$$V = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \\ L_{a_9} \\ L_{a_{10}} \\ \hline L_{a_{11}} \\ L_{a_{12}} \\ L_{a_{13}} \\ L_{a_{14}} \\ L_{a_{15}} \end{array}\right], \left[\begin{array}{c|cc|cccc} L_{a_1} & L_{a_3} & L_{a_5} & L_{a_7} & L_{a_9} & L_{a_{11}} & L_{a_{13}} \\ L_{a_2} & L_{a_4} & L_{a_6} & L_{a_8} & L_{a_{10}} & L_{a_{12}} & L_{a_{14}} \end{array}\right], \right.$$

$$\left. \left[\begin{array}{c|ccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array}\right], \left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ \hline L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \end{array}\right] \;\middle|\; \begin{array}{c} L_{a_i} \in L_{Q^+ \cup \{0\}}; \\ 1 \leq i \leq 18 \end{array} \right\}$$

be a set super matrix semivector space of refined labels over the set S = $\{3Z^+ \cup 7Z^+ \cup 13Z^+ \cup 11Z^+ \cup 19Z^+ \cup 8Z^+ \cup \{0\}\} \subseteq Q^+ \cup \{0\}$.

Consider

$$M_1 = \left\{ \left. \begin{bmatrix} \dfrac{L_{a_1}}{L_{a_2}} \\ \dfrac{L_{a_3}}{L_{a_4}} \\ L_{a_5} \\ \dfrac{L_{a_6}}{L_{a_7}} \\ L_{a_8} \\ L_{a_9} \\ \dfrac{L_{a_{10}}}{L_{a_{11}}} \\ L_{a_{12}} \\ L_{a_{13}} \\ L_{a_{14}} \\ L_{a_{15}} \end{bmatrix}, \left[\begin{array}{c|cc|cccc} L_{a_1} & 0 & 0 & 0 & 0 & 0 & L_{a_3} \\ L_{a_2} & 0 & 0 & 0 & 0 & 0 & L_{a_4} \end{array}\right] \right| \begin{array}{c} L_{a_i} \in L_{Q^+ \cup \{0\}}; \\ 1 \le i \le 15 \end{array} \right\}$$

$$\subseteq V,$$

$$M_2 = \left\{ \left. \left[\begin{array}{c|cc|cccc} L_{a_1} & L_{a_4} & L_{a_5} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ L_{a_2} & L_{a_3} & L_{a_6} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} \end{array}\right] \right| \begin{array}{c} L_{a_i} \in L_{Q^+ \cup \{0\}}; \\ 1 \le i \le 14 \end{array} \right\} \subseteq V,$$

$$M_3 = \left\{ \left. \left[\begin{array}{c|ccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array}\right], \left[\begin{array}{c|cc|cccc} L_{a_1} & L_{a_3} & 0 & 0 & 0 & 0 & L_{a_5} \\ L_{a_2} & L_{a_4} & 0 & 0 & 0 & 0 & L_{a_6} \end{array}\right] \right| \right.$$

$$\left. L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \le i \le 16 \right\} \subseteq V$$

and

$$M_4 = \left\{\left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ \hline L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \end{array}\right] \middle| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \le i \le 18 \right.$$

and

$$\left. \left[\begin{array}{c|cc|cccc} L_{a_1} & 0 & 0 & 0 & 0 & 0 & L_{a_5} \\ L_{a_2} & L_{a_3} & 0 & 0 & L_{a_4} & 0 & L_{a_6} \end{array}\right] \middle| L_{a_j} \in L_{Q^+ \cup \{0\}}; 1 \le j \le 16 \right\} \subseteq V$$

be set super matrix semivector subspaces of refined labels of V over the set $S \subseteq Q^+ \cup \{0\}$.

Clearly $V = \bigcup_{i=1}^{4} M_i$ but $M_i \cap M_j \neq \phi$ if $i \neq j$, $1 \le i, j \le 4$. Thus V is only a pseudo direct union of set super matrices semivector subspace of refined labels over S.

We can define set linear transformation of set super matrix semivector spaces only if both the set super matrix semivector space of refined labels are defined over the same set $S \subseteq R^+ \cup \{0\}$.

We will just illustrate this situation by some examples.

Example 5.56: Let

$$V = \left\{ \left[\begin{array}{c|cc} L_{a_1} & L_{a_8} & L_{a_{15}} \\ \hline L_{a_2} & L_{a_9} & L_{a_{16}} \\ L_{a_3} & L_{a_{10}} & L_{a_{17}} \\ \hline L_{a_4} & L_{a_{11}} & L_{a_{18}} \\ L_{a_5} & L_{a_{12}} & L_{a_{19}} \\ L_{a_6} & L_{a_{13}} & L_{a_{20}} \\ \hline L_{a_7} & L_{a_{14}} & L_{a_{21}} \end{array}\right], \left(L_{a_1} \;\; L_{a_2} \,\middle|\, L_{a_3} \;\; L_{a_4} \;\; L_{a_5} \,\middle|\, L_{a_6} \;\; L_{a_7} \,\middle|\, L_{a_8}\right), \right.$$

$$\left.\left\{\left[\begin{array}{cc|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ \hline L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ \hline L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \\ L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} & L_{a_{25}} \end{array}\right] \right| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \le i \le 25 \right\}$$

be a set super matrix semivector space of refined labels over S = a ∈ $5Z^+ \cup 3Z^+ \cup \{0\}\} \subseteq Q^+ \cup \{0\}$. Let

$$W = \left\{ \left(\begin{array}{c|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \end{array}\right), \left[\begin{array}{c|cc|cccc} L_{a_1} & L_{a_4} & L_{a_7} & L_{a_{10}} & L_{a_{13}} & L_{a_{16}} & L_{a_{19}} \\ L_{a_2} & L_{a_5} & L_{a_8} & L_{a_{11}} & L_{a_{14}} & L_{a_{17}} & L_{a_{20}} \\ L_{a_3} & L_{a_6} & L_{a_9} & L_{a_{12}} & L_{a_{15}} & L_{a_{18}} & L_{a_{21}} \end{array}\right], \right.$$

$$\left.\left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ \hline L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \\ L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \end{array}\right] \right| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \le i \le 24 \right\}$$

be a set super matrix semivector space of refined labels over S = {a ∈ 5Z+ ∪ 3Z+ ∪ {0}} ⊆ $Q^+ \cup \{0\}$. Define T : V → W a set linear transformation of refined label set super semivector spaces as follows.

$$T\left(\left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ \hline L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \end{array}\right]\right)$$

$$= \left[\begin{array}{c|cc|cccc} L_{a_1} & L_{a_4} & L_{a_7} & L_{a_{10}} & L_{a_{13}} & L_{a_{16}} & L_{a_{19}} \\ L_{a_2} & L_{a_5} & L_{a_8} & L_{a_{11}} & L_{a_{14}} & L_{a_{17}} & L_{a_{20}} \\ L_{a_3} & L_{a_6} & L_{a_9} & L_{a_{12}} & L_{a_{15}} & L_{a_{18}} & L_{a_{21}} \end{array}\right],$$

$$T\left(\left(L_{a_1} \quad L_{a_2} \mid L_{a_3} \quad L_{a_4} \quad L_{a_5} \mid L_{a_6} \quad L_{a_7} \mid L_{a_8}\right)\right) =$$

$$\left(\begin{array}{c|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \end{array}\right)$$

and

$$T\left(\left[\begin{array}{cc|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ \hline L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ \hline L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \\ L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} & L_{a_{25}} \end{array}\right]\right) = \left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ \hline L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \\ L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \end{array}\right].$$

T is a set super linear transformation of set super linear semivector spaces of refined labels defined over the set $S \subseteq Q^+ \cup \{0\}$.

Example 5.57: Let

$$V = \left\{\left[\begin{array}{ccc} L_{a_1} & L_{a_8} & L_{a_{15}} \\ \hline L_{a_2} & L_{a_9} & L_{a_{16}} \\ L_{a_3} & L_{a_{10}} & L_{a_{17}} \\ \hline L_{a_4} & L_{a_{11}} & L_{a_{18}} \\ L_{a_5} & L_{a_{12}} & L_{a_{19}} \\ L_{a_6} & L_{a_{13}} & L_{a_{20}} \\ L_{a_7} & L_{a_{14}} & L_{a_{21}} \end{array}\right], \left[\begin{array}{c|cc|c|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ \hline L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ \hline L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \\ \hline L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} & L_{a_{25}} \end{array}\right],\right.$$

$$\left.\left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ \hline \vdots \\ \hline L_{a_{23}} \\ L_{a_{24}} \\ L_{a_{25}} \end{array}\right], \left[\begin{array}{c|cc|cccc} L_{a_1} & L_{a_4} & L_{a_7} & L_{a_{10}} & L_{a_{13}} & L_{a_{16}} & L_{a_{19}} \\ L_{a_2} & L_{a_5} & L_{a_8} & L_{a_{11}} & L_{a_{14}} & L_{a_{17}} & L_{a_{20}} \\ L_{a_3} & L_{a_6} & L_{a_9} & L_{a_{12}} & L_{a_{15}} & L_{a_{18}} & L_{a_{21}} \end{array}\right] \right| \begin{array}{c} L_{a_i} \in L_{Q^+ \cup \{0\}}; \\ 1 \leq i \leq 25 \end{array} \right\}$$

be a set super matrix semivector space of refined labels over the set S = $\{a \in 3Z+ \cup \{0\} \cup 2Z+\}$.

Define a map $T : V \to V$ given by

$$T\left(\begin{bmatrix} L_{a_1} & L_{a_8} & L_{a_{15}} \\ L_{a_2} & L_{a_9} & L_{a_{16}} \\ L_{a_3} & L_{a_{10}} & L_{a_{17}} \\ L_{a_4} & L_{a_{11}} & L_{a_{18}} \\ L_{a_5} & L_{a_{12}} & L_{a_{19}} \\ L_{a_6} & L_{a_{13}} & L_{a_{20}} \\ L_{a_7} & L_{a_{14}} & L_{a_{21}} \end{bmatrix}\right)$$

$$= \left[\begin{array}{c|cc|cccc} L_{a_1} & L_{a_4} & L_{a_7} & L_{a_{10}} & L_{a_{13}} & L_{a_{16}} & L_{a_{19}} \\ L_{a_2} & L_{a_5} & L_{a_8} & L_{a_{11}} & L_{a_{14}} & L_{a_{17}} & L_{a_{20}} \\ L_{a_3} & L_{a_6} & L_{a_9} & L_{a_{12}} & L_{a_{15}} & L_{a_{18}} & L_{a_{21}} \end{array}\right]$$

$$T\left(\left[\begin{array}{c|cc|c|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ \hline L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ \hline L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \\ \hline L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} & L_{a_{25}} \end{array}\right]\right) = \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ \hline \vdots \\ \hline L_{a_{23}} \\ L_{a_{24}} \\ L_{a_{25}} \end{array}\right],$$

$$T\left(\left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ \hline \vdots \\ \hline L_{a_{23}} \\ L_{a_{24}} \\ L_{a_{25}} \end{array}\right]\right) = \left[\begin{array}{c|cc|c|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ \hline L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ \hline L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \\ \hline L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} & L_{a_{25}} \end{array}\right]$$

and

$$T\left(\left[\begin{array}{c|cc|cccc} L_{a_1} & L_{a_4} & L_{a_7} & L_{a_{10}} & L_{a_{13}} & L_{a_{16}} & L_{a_{19}} \\ L_{a_2} & L_{a_5} & L_{a_8} & L_{a_{11}} & L_{a_{14}} & L_{a_{17}} & L_{a_{20}} \\ L_{a_3} & L_{a_6} & L_{a_9} & L_{a_{12}} & L_{a_{15}} & L_{a_{18}} & L_{a_{21}} \end{array}\right]\right)$$

$$= \left[\begin{array}{ccc} L_{a_1} & L_{a_8} & L_{a_{15}} \\ \hline L_{a_2} & L_{a_9} & L_{a_{16}} \\ L_{a_3} & L_{a_{10}} & L_{a_{17}} \\ \hline L_{a_4} & L_{a_{11}} & L_{a_{18}} \\ L_{a_5} & L_{a_{12}} & L_{a_{19}} \\ L_{a_6} & L_{a_{13}} & L_{a_{20}} \\ L_{a_7} & L_{a_{14}} & L_{a_{21}} \end{array}\right].$$

Clearly T is a set super matrix semivector space set linear operator of refined labels.

As in case of usual vector spaces we can talk of a set super matrix semivector subspace of refined labels which is invariant under a set super linear operator T of V.

Example 5.58: Let

$$V = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \\ \hline L_{a_9} \end{array}\right], \left[\begin{array}{c|c|c|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ \hline L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array}\right], \left(\begin{array}{c|c} L_{a_1} & L_{a_3} \\ \hline L_{a_2} & L_{a_4} \end{array}\right), \right.$$

$$\left. \left[\begin{array}{ccc|cc|c|c} L_{a_1} & L_{a_4} & L_{a_7} & L_{a_{10}} & L_{a_{13}} & L_{a_{16}} & L_{a_{19}} \\ L_{a_2} & L_{a_5} & L_{a_8} & L_{a_{11}} & L_{a_{14}} & L_{a_{17}} & L_{a_{20}} \\ L_{a_3} & L_{a_6} & L_{a_9} & L_{a_{12}} & L_{a_{15}} & L_{a_{18}} & L_{a_{21}} \end{array}\right] \;\middle|\; L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \le i \le 21 \right\}$$

be a set super matrix semivector space of refined labels over the set S = a ∈ $3Z^+ \cup 2Z^+ \cup 7Z^+ \cup \{0\}\} \subseteq Q^+ \cup \{0\}$.

Consider

$$W = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \\ \hline L_{a_9} \end{array}\right], \left(\begin{array}{c|c} L_{a_1} & L_{a_3} \\ \hline L_{a_2} & L_{a_4} \end{array}\right), \left[\begin{array}{c|c|c|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ \hline L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array}\right] \middle| \begin{array}{c} L_{a_i} \in L_{Q^+ \cup \{0\}}; \\ 1 \leq i \leq 16 \end{array} \right\}$$

$\subseteq$ V is a set super matrix semivector subspace of V of refined labels over the set S. Define a set super linear transformation $T : V \to V$ by

$$T\left(\left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \\ \hline L_{a_9} \end{array}\right]\right) = \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \\ \hline L_{a_9} \end{array}\right]$$

$$T\left(\left(\begin{array}{c|c} L_{a_1} & L_{a_3} \\ \hline L_{a_2} & L_{a_4} \end{array}\right)\right) = \left(\begin{array}{c|c} L_{a_1} & L_{a_3} \\ \hline L_{a_2} & L_{a_4} \end{array}\right),$$

$$T\left(\left[\begin{array}{c|c|c|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ \hline L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array}\right]\right) = \left[\begin{array}{c|c|c|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ \hline L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array}\right]$$

and

$$T\left(\left[\begin{array}{ccc|cc|c|c} L_{a_1} & L_{a_4} & L_{a_7} & L_{a_{10}} & L_{a_{13}} & L_{a_{16}} & L_{a_{19}} \\ L_{a_2} & L_{a_5} & L_{a_8} & L_{a_{11}} & L_{a_{14}} & L_{a_{17}} & L_{a_{20}} \\ L_{a_3} & L_{a_6} & L_{a_9} & L_{a_{12}} & L_{a_{15}} & L_{a_{18}} & L_{a_{21}} \end{array}\right]\right) =$$

$$\left(\begin{array}{ccc|cc|c|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

T is a set linear super operator on V.

Further T (W) $\subseteq$ W. Thus W is invariant under the set linear super operator of V.

Now we proceed onto define the notion of semigroup super matrix semivector space of refined labels over a semigroup S $\subseteq$ $R^+ \cup \{0\}$.

DEFINITION 5.7: *Let V be a set super matrix semivector space of refined labels over the set S $\subseteq R^+ \cup \{0\}$.*

If S is a semigroup of refined labels contained in $R^+ \cup \{0\}$ then we call the set super matrix semivector space of refined labels as semigroup super matrix semivector space of refined labels over the semigroup S $\subseteq R^+ \cup \{0\}$.

We will illustrate this situation by some examples.

Example 5.59: Let

$$V = \left\{ \left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} \\ L_{a_{15}} & L_{a_{16}} \\ \hline L_{a_{17}} & L_{a_{18}} \end{array}\right], \left[\begin{array}{cc|ccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ \hline L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ \hline L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \end{array}\right], \right.$$

$$\left. \left(L_{a_1} \quad L_{a_2} \mid L_{a_3} \quad L_{a_4} \mid L_{a_5} \mid L_{a_6} \quad L_{a_7}\right) \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 20 \right\}$$

be a semigroup super matrix semivector space of refined labels over the semigroup $S = 3Z^+ \cup \{0\}$.

Example 5.60: Let

$$M = \left\{ \left[\begin{array}{c|c} L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} \end{array}\right], \left(L_{a_1} \quad L_{a_2} \mid L_{a_3} \quad L_{a_4} \quad L_{a_5} \mid L_{a_6} \quad L_{a_7} \mid L_{a_8}\right), \right.$$

$$\left. \left[\begin{array}{cc|ccc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ \hline L_{a_{19}} & L_{a_{20}} & L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ L_{a_{25}} & L_{a_{26}} & L_{a_{27}} & L_{a_{28}} & L_{a_{29}} & L_{a_{30}} \end{array}\right] \middle| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \le i \le 30 \right\}$$

is a semigroup super matrix semivector space of refined labels over the semigroup $Z^+ \cup \{0\}$ under addition.

It is pertinent to mention here that the semigrioup must be only a subset of $R^+ \cup \{0\}$ but it can be a semigroup under addition or under multiplication.

Example 5.61: Let

$$K = \left\{ \left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ \hline L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \\ L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ L_{a_{25}} & L_{a_{26}} & L_{a_{27}} \\ L_{a_{28}} & L_{a_{29}} & L_{a_{30}} \end{array}\right], \left[\begin{array}{cc|c|ccc} L_{a_1} & L_{a_5} & L_{a_9} & L_{a_{13}} & L_{a_{17}} & L_{a_{21}} \\ L_{a_2} & L_{a_6} & L_{a_{10}} & L_{a_{14}} & L_{a_{18}} & L_{a_{22}} \\ L_{a_3} & L_{a_7} & L_{a_{11}} & L_{a_{15}} & L_{a_{19}} & L_{a_{23}} \\ L_{a_4} & L_{a_8} & L_{a_{12}} & L_{a_{16}} & L_{a_{20}} & L_{a_{24}} \end{array}\right], \right.$$

$$\left. \left[\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right] \right| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \leq i \leq 30 \left. \vphantom{\begin{array}{c} L \\ L \\ L \\ L \end{array}} \right\}$$

be a semigroup super matrix semivector space over the semigroup $L_{Q^+ \cup \{0\}}$ under addition.

Example 5.62: Let

$$T = \left\{ \left[\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \end{array}\right], \left(L_{a_1} \mid L_{a_2} \;\; L_{a_3} \;\; L_{a_4} \mid L_{a_5} \;\; L_{a_6} \mid L_{a_7}\right), \right.$$

$$\left. \left[\begin{array}{c|ccc|cc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ \hline L_{a_{19}} & L_{a_{20}} & L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ L_{a_{25}} & L_{a_{26}} & L_{a_{27}} & L_{a_{28}} & L_{a_{29}} & L_{a_{30}} \end{array}\right] \;\middle|\; L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \leq i \leq 30 \right\}$$

be a semigroup super matrix semivector space of refined labels over $S = L_{R^+ \cup \{0\}}$. S a semigroup under multiplication.

We can define substructure which is left as an exercise to the reader.

However we give some examples of them.

Example 5.63: Let

$$V = \left\{ \left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ \hline L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \\ L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ \hline L_{a_{25}} & L_{a_{26}} & L_{a_{27}} \\ L_{a_{28}} & L_{a_{29}} & L_{a_{30}} \end{array}\right], \left[\begin{array}{ccc|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right], \right.$$

$$\left\{\left(\begin{array}{c|c} L_{a_1} & L_{a_3} \\ \hline L_{a_2} & L_{a_4} \end{array}\right), \left[\begin{array}{c|cc|ccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ \hline L_{a_{19}} & L_{a_{20}} & L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ L_{a_{25}} & L_{a_{26}} & L_{a_{27}} & L_{a_{28}} & L_{a_{29}} & L_{a_{30}} \end{array}\right] \middle| \begin{array}{c} L_{a_i} \in L_{Q^+\cup\{0\}}; \\ 1 \leq i \leq 30 \end{array}\right\}$$

be a semigroup super matrix semivector space of refined labels over the semigroup $S = L_{Q^+\cup\{0\}}$. Take

$$W = \left\{\left[\begin{array}{ccc|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right], \left(\begin{array}{c|c} L_{a_1} & L_{a_3} \\ \hline L_{a_2} & L_{a_4} \end{array}\right) \middle| \begin{array}{c} L_{a_i} \in L_{R^+\cup\{0\}}; \\ 1 \leq i \leq 12 \end{array}\right\} \subseteq V,$$

W is a semigroup super matrix semivector subspace of refined labels over the semigroup $S = L_{Q^+\cup\{0\}}$. Take

$$B = \left\{\left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ \hline L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \\ L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ \hline L_{a_{25}} & L_{a_{26}} & L_{a_{27}} \\ L_{a_{28}} & L_{a_{29}} & L_{a_{30}} \end{array}\right], \left(\begin{array}{c|c} L_{a_1} & L_{a_3} \\ \hline L_{a_2} & L_{a_4} \end{array}\right) \middle| \begin{array}{c} L_{a_i} \in L_{R^+\cup\{0\}}; \\ 1 \leq i \leq 30 \end{array}\right\} \subseteq V$$

is a semigroup super matrix semivector subspace of refined labels over the semigroup S.

Clearly

$$B \cap W = \left\{\left(\begin{array}{c|c} L_{a_1} & L_{a_3} \\ \hline L_{a_2} & L_{a_4} \end{array}\right) \middle| L_{a_i} \in L_{R^+\cup\{0\}}; 1 \leq i \leq 4\right\} \subseteq V$$

is again a semigroup super matrix semivector subspace of refined labels over the semigroup S.

Example 5.64: Let

$$M = \left\{ \left[\begin{array}{c|cc} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \end{array}\right], \left[\begin{array}{c|cc|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \end{array}\right], \right.$$

$$\left. \left[\begin{array}{c|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array}\right] \right| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \le i \le 18 \right\}$$

be a semigroup super matrix semivector space of refined labels over the semigroup S = $L_{Q^+ \cup \{0\}}$ under multiplication. Consider

$$W_1 = \left\{ \left[\begin{array}{c|cc} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \end{array}\right] \right| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \le i \le 15 \right\} \subseteq V,$$

$$W_2 = \left\{ \left[\begin{array}{c|cc|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \end{array}\right] \right| \begin{array}{c} L_{a_i} \in L_{Q^+ \cup \{0\}}; \\ 1 \le i \le 18 \end{array} \right\} \subseteq V$$

and $$W_3 = \left\{ \left[\begin{array}{c|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array}\right] \right| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \le i \le 16 \right\}$$

$\subseteq$ V be semigroup super matrix semivector subspaces of V of refined labels over S. Clearly $W_i \cap W_j = \phi$ if $i \neq j$, $1 \leq i, j \leq 3$. Also $V = \bigcup_{i=1}^{3} W_i$. Thus is a direct union of semigroup super matrix semivector subspaces of refined labels over $L_{R^+ \cup \{0\}} = S$.

Example 5.65: Let

$$M = \left\{ \left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} \end{array}\right], \left[\begin{array}{c|ccc|cc} L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ \hline L_{a_2} & L_{a_2} & L_{a_2} & L_{a_2} & L_{a_2} & L_{a_2} \\ L_{a_3} & L_{a_3} & L_{a_3} & L_{a_3} & L_{a_3} & L_{a_3} \\ L_{a_4} & L_{a_4} & L_{a_4} & L_{a_4} & L_{a_4} & L_{a_4} \\ \hline L_{a_5} & L_{a_5} & L_{a_5} & L_{a_5} & L_{a_5} & L_{a_5} \end{array}\right], \right.$$

$$\left. \left(L_{a_1} \,|\, L_{a_2} \;\; L_{a_1} \;\; L_{a_2} \,|\, L_{a_1} \;\; L_{a_2} \;\; L_{a_1} \;\; L_{a_2}\right) \,\middle|\, L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 14 \right\}$$

be a semigroup super matrix semivector space of refined labels over the semigroup $S = L_{Q^+ \cup \{0\}}$.

Consider

$$P_1 = \left\{ \left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} \end{array}\right], \left[\begin{array}{c|ccc|cc} L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline L_{a_5} & L_{a_5} & L_{a_5} & L_{a_5} & L_{a_5} & L_{a_5} \end{array}\right] \right|$$

$$\left. L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 14 \right\} \subseteq M,$$

$$P_2 = \left\{ \left[\begin{array}{c|ccc|cc} L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ \hline L_{a_2} & L_{a_2} & L_{a_2} & L_{a_2} & L_{a_2} & L_{a_2} \\ L_{a_3} & L_{a_3} & L_{a_3} & L_{a_3} & L_{a_3} & L_{a_3} \\ L_{a_4} & L_{a_4} & L_{a_4} & L_{a_4} & L_{a_4} & L_{a_4} \\ \hline L_{a_5} & L_{a_5} & L_{a_5} & L_{a_5} & L_{a_5} & L_{a_5} \end{array}\right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 5 \right\}$$

$$\subseteq M$$

and

$$P_3 = \left\{ \left(L_{a_1} \mid L_{a_2} \quad L_{a_1} \quad L_{a_2} \mid L_{a_1} \quad L_{a_2} \quad L_{a_1} \quad L_{a_2}\right), \right.$$

$$\left. \left[\begin{array}{c|ccc|cc} L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 2 \right\} \subseteq M$$

be semigroup super matrix semivector subspaces of M.

We see

$$P_1 \cap P_2 =$$

$$\left\{ \left[\begin{array}{c|ccc|cc} L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline L_{a_5} & L_{a_5} & L_{a_5} & L_{a_5} & L_{a_5} & L_{a_5} \end{array}\right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; i = 1,5 \right\} \subseteq M$$

is again a semigroup super matrix semivector subspace of M.

$$P_2 \cap P_3 = \left\{ \left[\begin{array}{c|ccc|cc} L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \middle| L_{a_1} \in L_{R^+ \cup \{0\}} \right\} \subseteq M$$

and

$$P_3 \cap P_1 =$$

$$\left\{ \left[\begin{array}{c|ccc|cc} L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \middle| L_{a_1} \in L_{R^+ \cup \{0\}} \right\} \subseteq M$$

are again semigroup super matrix semivector subspaces of M over the semigroup $S = L_{Q^+ \cup \{0\}}$.

We see thus $P_i \cap P_j \neq \phi$; if $i \neq j$; $1 \leq i, j \leq 3$. But $M = \bigcup_{i=1}^{3} P_i$ so M is a pseudo direct union of semigroup super matrix semivector subspaces of M over S.

Example 5.66: Let

$$P = \left\{ \left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \end{array} \right], \left(L_{a_1} \,|\, L_{a_2} \,|\, L_{a_3} \;\; L_{a_4} \;\; L_{a_5} \,|\, L_{a_6} \;\; L_{a_7} \,|\, L_{a_8} \right), \right.$$

$$\left\{\left[\begin{array}{c|cc|ccc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} \\ \hline L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} \\ L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} \\ L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_7} & L_{a_6} \\ \hline L_{a_7} & L_{a_3} & L_{a_2} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_1} \end{array}\right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 21 \right\}$$

be a semigroup of super matrix semivector space of refined labels over $L_{R^+ \cup \{0\}} = S$.

Consider

$$M = \left\{\left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \end{array}\right], \left(L_{a_1} \,|\, L_{a_1} \,|\, L_{a_1} \;\; L_{a_1} \;\; L_{a_1} \,|\, L_{a_1} \;\; L_{a_1} \,|\, L_{a_1}\right) \right|$$

$L_{a_i} \in L_{R^+ \cup \{0\}}\Big\} \subseteq P$ and $T = \{ L_{Q^+ \cup \{0\}} \} \subseteq S$ be a subsemigroup of S.

We see M is a semigroup super matrix semivector space of refined labels over T of P called the subsemigroup super matrix semivector subspace of refined labels over the subsemigroup T of S.

Now we can give the related definition.

DEFINITION 5.8: *Let V be a semigroup super matrix semivector space of refined labels over the semigroup S. Let W ($\subseteq$ V) and P ($\subseteq$ S) be subsets of V and S respectively, if P be a subsemigroup of S and W is a semigroup super matrix semivector space over the semigroup P of refined labels then we*

define W to be a subsemigroup of super matrix semivector subspace of V refined labels over the subsemigroup P of the semigroup S. If V has no subsemigroup super matrix semivector subspace then we define V to be a pseudo simple semigroup super matrix semivector space of refined labels over the semigroup S.

Interested reader can supply examples of them.

We can define linear transformation of semigroup super matrix semivector spaces of refined labels provided they are defined over the same semigroup. This task is let to the reader, however we give some examples of them.

Example 5.67: Let

$$V = \left\{ \left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \end{array}\right], \left(L_{a_1} \; L_{a_2} \; L_{a_3} \,|\, L_{a_4} \; L_{a_5} \,|\, L_{a_6} \,|\, L_{a_7} \; L_{a_8} \; L_{a_9} \; L_{a_{10}}\right), \right.$$

$$\left. \left[\begin{array}{cc|ccc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_3} & L_{a_1} & L_{a_2} & L_{a_2} & L_{a_3} & L_{a_1} \\ L_{a_1} & L_{a_2} & L_{a_3} & L_{a_3} & L_{a_2} & L_{a_1} \\ \hline L_{a_2} & L_{a_3} & L_{a_1} & L_{a_1} & L_{a_3} & L_{a_2} \\ L_{a_1} & L_{a_2} & L_{a_1} & L_{a_2} & L_{a_3} & L_{a_3} \\ L_{a_2} & L_{a_2} & L_{a_3} & L_{a_2} & L_{a_3} & L_{a_1} \end{array}\right] \right| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 18 \right\}$$

be a semigroup super matrix semivector space of refined labels over the semigroup $S = L_{Q^+ \cup \{0\}}$.

$$W = \left\{ \left[\begin{array}{c|ccc|cc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ \hline L_{a_{19}} & L_{a_{20}} & L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ L_{a_{25}} & L_{a_{26}} & L_{a_{27}} & L_{a_{28}} & L_{a_{29}} & L_{a_{30}} \\ L_{a_{31}} & L_{a_{32}} & L_{a_{33}} & L_{a_{34}} & L_{a_{35}} & L_{a_{36}} \end{array}\right], \right.$$

$$\left(\begin{array}{c|cc|ccc|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \end{array}\right),$$

$$\left. \left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} \\ \hline L_{a_9} & L_{a_{10}} \end{array}\right] \right| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 36 \right\}$$

be a semigroup super matrix semivector space of refined labels over the same semigroup $S = L_{Q^+ \cup \{0\}}$.

Define $T : V \to W$ as follows:

$$T\left(\left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \end{array}\right]\right) =$$

$$\left(\begin{array}{c|cc|ccc|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \end{array}\right),$$

$$T\left(\left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \left| L_{a_4} \quad L_{a_5} \right| L_{a_6} \left| L_{a_7} \quad L_{a_8} \quad L_{a_9} \quad L_{a_{10}}\right.\right)\right) =$$

$$\left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} \\ \hline L_{a_9} & L_{a_{10}} \end{array}\right]$$

and

$$T\left(\left[\begin{array}{cc|ccc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_3} & L_{a_1} & L_{a_2} & L_{a_2} & L_{a_3} & L_{a_1} \\ L_{a_1} & L_{a_2} & L_{a_3} & L_{a_3} & L_{a_2} & L_{a_1} \\ \hline L_{a_2} & L_{a_3} & L_{a_1} & L_{a_1} & L_{a_3} & L_{a_2} \\ L_{a_1} & L_{a_2} & L_{a_1} & L_{a_2} & L_{a_3} & L_{a_3} \\ L_{a_2} & L_{a_2} & L_{a_3} & L_{a_2} & L_{a_3} & L_{a_1} \end{array}\right]\right)$$

$$= \left[\begin{array}{cc|ccc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_3} & L_{a_3} & L_{a_1} & L_{a_3} & L_{a_1} & L_{a_2} \\ L_{a_1} & L_{a_2} & L_{a_2} & L_{a_3} & L_{a_3} & L_{a_1} \\ \hline L_{a_2} & L_{a_3} & L_{a_1} & L_{a_1} & L_{a_2} & L_{a_2} \\ L_{a_1} & L_{a_1} & L_{a_3} & L_{a_3} & L_{a_1} & L_{a_1} \\ L_{a_3} & L_{a_3} & L_{a_2} & L_{a_1} & L_{a_3} & L_{a_2} \end{array}\right].$$

It is easily verified T is a semigroup super matrix semivector space linear transformation of refined labels over the semigroup S.

Example 5.68: Let

$$V = \left\{ \left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ \hline L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \\ L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ L_{a_{25}} & L_{a_{26}} & L_{a_{27}} \\ L_{a_{28}} & L_{a_{29}} & L_{a_{30}} \end{array}\right], \left(\begin{array}{c|cc|ccc|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \end{array}\right), \right.$$

$$\left. \left[\begin{array}{ccc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array}\right] \right| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 30 \right\}$$

be a semigroup super matrix semivector space of refined labels over the semigroup S = $L_{Q^+ \cup \{0\}}$. Define a map T : V → V

$$T\left(\left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ \hline L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \\ L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ L_{a_{25}} & L_{a_{26}} & L_{a_{27}} \\ L_{a_{28}} & L_{a_{29}} & L_{a_{30}} \end{array}\right]\right) = \left[\begin{array}{ccc|c} L_{a_1} & L_{a_3} & L_{a_5} & L_{a_7} \\ L_{a_9} & L_{a_{11}} & L_{a_{13}} & L_{a_{15}} \\ L_{a_{17}} & L_{a_{19}} & L_{a_{21}} & L_{a_{23}} \\ \hline L_{a_{25}} & L_{a_{27}} & L_{a_{29}} & L_{a_{30}} \end{array}\right],$$

$$T\left(\left(\begin{array}{c|cc|ccc|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \end{array}\right)\right)$$

$$= \begin{bmatrix} L_{a_1} & 0 & L_{a_3} \\ L_{a_2} & L_{a_{10}} & L_{a_6} \\ L_{a_3} & L_{a_{11}} & L_{a_9} \\ L_{a_4} & L_{a_{12}} & L_{a_{12}} \\ L_{a_5} & L_{a_{13}} & L_{a_{15}} \\ L_{a_6} & L_{a_{14}} & L_{a_{18}} \\ L_{a_7} & L_{a_{15}} & 0 \\ L_{a_8} & L_{a_{16}} & L_{a_1} \\ L_{a_9} & L_{a_{17}} & L_{a_2} \\ 0 & L_{a_{18}} & L_{a_3} \end{bmatrix}$$

and

$$T\left(\left[\begin{array}{ccc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array}\right]\right) =$$

$$\left(\begin{array}{c|cc|ccc|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} & 0 \\ 0 & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array}\right).$$

It is easily verified T is a semigroup super matrix semivector space linear operator on V.

Now we proceed onto give examples of the notion of integer semigroup super matrix semivector space of refined labels.

Example 5.69: Let

$$V = \left\{ \left[\begin{array}{c|cc} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \end{array}\right], \left(\begin{array}{c|cc|c|ccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} \\ L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} \end{array}\right), \right.$$

$$\left. \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ \hline L_{a_8} \\ \hline L_{a_9} \\ L_{a_{10}} \\ \hline L_{a_{11}} \end{array}\right] \right| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 14 \left. \vphantom{\begin{array}{c} 1\\1\\1\\1\\1\\1\\1\\1\\1\\1\\1 \end{array}} \right\}$$

be an integer group super matrix semivector space of refined labels over the semigroup $S = Z^+ \cup \{0\}$.

Example 5.70: Let

$$V = \left\{ \left[\begin{array}{cc} L_{a_1} & L_{a_{12}} \\ \hline L_{a_2} & L_{a_{13}} \\ L_{a_3} & L_{a_{14}} \\ L_{a_4} & L_{a_{15}} \\ L_{a_5} & L_{a_{16}} \\ \hline L_{a_6} & L_{a_{17}} \\ L_{a_7} & L_{a_{18}} \\ L_{a_8} & L_{a_{19}} \\ \hline L_{a_9} & L_{a_{20}} \\ L_{a_{10}} & L_{a_{21}} \\ \hline L_{a_{11}} & L_{a_{22}} \end{array}\right], \left[\begin{array}{c|cc|cc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ \hline L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \\ \hline L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} & L_{a_{25}} \end{array}\right], \right.$$

$$\left. \left[\begin{array}{c|c} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} \end{array}\right] \right| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 25 \Bigg\}$$

be an integer semigroup super matrix semivector space of refined labels over the semigroup $3Z^+ \cup \{0\}$.

Substructures, linear transformation can be defined as in case of usual semigroup super matrix semivector spaces of refined labels.

Now we proceed onto define the notion of group super matrix semivector spaces of refined labels over a group.

DEFINITION 5.9: *Let V be a semigroup super matrix semivector space of refined over the semigroup $S = L_{Q^+}$; (L_{R^+}) under multiplication. If S is a group under multiplication then we*

define V to be a group super matrix semivector space of refined labels over the multiplication group L_{Q^+} *or* L_{R^+}.

We will first illustrate this situation by some examples.

Example 5.71: Let

$$V = \left\{ \left[\begin{array}{c|c} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} \end{array}\right], \left[\begin{array}{cc|c|cccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} \\ L_{a_2} & L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} \\ \hline L_{a_3} & L_{a_9} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ \hline L_{a_4} & L_{a_{10}} & L_{a_{15}} & L_{a_{19}} & L_{a_{20}} & L_{a_{21}} & L_{a_{22}} \\ L_{a_5} & L_{a_{11}} & L_{a_{16}} & L_{a_{20}} & L_{a_{23}} & L_{a_{24}} & L_{a_{25}} \\ L_{a_6} & L_{a_{12}} & L_{a_{17}} & L_{a_{21}} & L_{a_{24}} & L_{a_{26}} & L_{a_{27}} \\ L_{a_7} & L_{a_{13}} & L_{a_{18}} & L_{a_{22}} & L_{a_{25}} & L_{a_{27}} & L_{a_{28}} \end{array}\right], \right.$$

$$\left. \left[\begin{array}{ccc|c|cc|c|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \end{array}\right] \;\middle|\; L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 28 \right\}$$

be a group super matrix semivector space of refined labels over the multiplicative group L_{Q^+}.

Example 5.70: Let

$$M = \left\{ \left[\begin{array}{cc} L_{a_1} & L_{a_{12}} \\ \hline L_{a_2} & L_{a_{13}} \\ L_{a_3} & L_{a_{14}} \\ \hline L_{a_4} & L_{a_{15}} \\ L_{a_5} & L_{a_{16}} \\ L_{a_6} & L_{a_{17}} \\ \hline L_{a_7} & L_{a_{18}} \\ L_{a_8} & L_{a_{19}} \\ L_{a_9} & L_{a_{20}} \\ L_{a_{10}} & L_{a_{21}} \end{array}\right], \left[\begin{array}{c|cc|ccc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} \\ L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} \end{array}\right], \right.$$

$$\left.\begin{bmatrix} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ L_{a_{19}} & L_{a_{20}} & L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} & L_{a_{25}} & L_{a_{26}} & L_{a_{27}} \\ L_{a_{28}} & L_{a_{29}} & L_{a_{30}} & L_{a_{31}} & L_{a_{32}} & L_{a_{33}} & L_{a_{34}} & L_{a_{35}} & L_{a_{36}} \\ L_{a_{37}} & L_{a_{38}} & L_{a_{39}} & L_{a_{40}} & L_{a_{41}} & L_{a_{42}} & L_{a_{43}} & L_{a_{44}} & L_{a_{45}} \\ L_{a_{46}} & L_{a_{47}} & L_{a_{48}} & L_{a_{49}} & L_{a_{50}} & L_{a_{51}} & L_{a_{52}} & L_{a_{53}} & L_{a_{54}} \end{bmatrix}\right|$$

$$\left. L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 54 \right\}$$

be a group super matrix semivector space of refined labels over L_{R^+} the group of positive refined labels.

We can define as a matter of routine the substructures in them.

Here we present them by some examples.

Example 5.73: Let

$$V = \left\{ \left[\begin{array}{c|cc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right], \left(L_{a_1} \;\; L_{a_2} \;\; L_{a_3} \;\; L_{a_4} \;\middle|\; L_{a_5} \;\; L_{a_6} \;\; L_{a_7} \;\middle|\; L_{a_8} \;\; L_{a_9} \;\middle|\; L_{a_{10}} \right), \right.$$

$$\left. \left[\begin{array}{c|c} L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \end{array}\right], \left[\begin{array}{cc|c|cc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \end{array}\right] \right| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 12 \left. \vphantom{\begin{bmatrix} L \\ L \end{bmatrix}} \right\}$$

be a group super matrix semivector space of refined labels over the group $G = L_{Q^+}$.

Now consider

$$P = \left\{ \left[\begin{array}{c|cc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right], \left[\begin{array}{c|c} L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \end{array}\right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 12 \right\}$$

$\subseteq$ V is a group super matrix semivector subspace of refined labels over the group G = L_{Q^+} under multiplication.

Example 5.72: Let

$$V = \left\{ \left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} \\ \hline L_{a_9} & L_{a_{10}} \end{array}\right], \right.$$

$$\left[\begin{array}{c|cc|ccc|c|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array}\right],$$

$$\left. \left[\begin{array}{c|ccc|cc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_2} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} \\ L_{a_3} & L_{a_8} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_4} & L_{a_9} & L_{a_{13}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ \hline L_{a_5} & L_{a_{10}} & L_{a_{14}} & L_{a_{17}} & L_{a_{19}} & L_{a_{20}} \\ L_{a_6} & L_{a_{11}} & L_{a_{15}} & L_{a_{18}} & L_{a_{20}} & L_{a_{21}} \end{array}\right] \middle| \begin{array}{c} L_{a_i} \in L_{R^+ \cup \{0\}}; \\ 1 \leq i \leq 21 \end{array} \right\}$$

be a group super matrix semivector space of refined labels over the group G = L_{Q^+}, L_{Q^+} group under multiplication.

Consider

$$P_1 = \left\{ \left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} \\ \hline L_{a_9} & L_{a_{10}} \end{array}\right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 10 \right\} \subseteq V,$$

$$P_2 = \left\{ \left[\begin{array}{c|cc|ccc|c|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array}\right] \middle| \begin{array}{c} L_{a_i} \in L_{R^+ \cup \{0\}}; \\ 1 \le i \le 16 \end{array} \right\} \subseteq V$$

and

$$P_3 = \left\{ \left[\begin{array}{c|ccc|cc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_2} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} \\ L_{a_3} & L_{a_8} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_4} & L_{a_9} & L_{a_{13}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ \hline L_{a_5} & L_{a_{10}} & L_{a_{14}} & L_{a_{17}} & L_{a_{19}} & L_{a_{20}} \\ L_{a_6} & L_{a_{11}} & L_{a_{15}} & L_{a_{18}} & L_{a_{20}} & L_{a_{21}} \end{array}\right] \middle| \begin{array}{c} L_{a_i} \in L_{R^+ \cup \{0\}}; \\ 1 \le i \le 21 \end{array} \right\} \subseteq V$$

be group super matrix semivector subspaces of refined labels of V over the multiplicative group L_{Q^+}.

We see

$$V = \bigcup_{i=1}^{3} P_i \; ; \; P_i \cap P_j = \phi \text{ if } i \ne j, \; 1 \le i, j \le 3,$$

thus V is a direct sum of group super matrix semivector subspaces of refined labels of V.

Example 5.75: Let

$$V = \left\{ \left[\begin{array}{c|c} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \end{array}\right], \left[\begin{array}{ccc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array}\right], \right.$$

$$\left[\begin{array}{c|c|c|cc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \end{array}\right]$$

$$\left.\left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} \\ L_{a_{15}} & L_{a_{16}} \\ \hline L_{a_{17}} & L_{a_{18}} \end{array}\right], \left(L_{a_1} \mid L_{a_2} \quad L_{a_3} \mid L_{a_4} \quad L_{a_5} \mid L_{a_6} \mid L_{a_7}\right) \right|$$

$L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 20 \}$ be a group super matrix semivector space of refined labels over the group $G = L_{Q^+}$, $L_{Q^+} = G$ is a multiplicative group.

Consider

$$H_1 = \left\{ \left[\begin{array}{c|c} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \end{array} \right], \left(L_{a_1} \,|\, L_{a_2} \;\; L_{a_3} \,|\, L_{a_4} \;\; L_{a_5} \,|\, L_{a_6} \,|\, L_{a_7} \right) \,\middle|\, \begin{array}{c} L_{a_i} \in L_{R^+ \cup \{0\}}; \\ 1 \le i \le 7 \end{array} \right\} \subseteq V,$$

$$H_2 = \left\{ \left[\begin{array}{ccc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array} \right], \right.$$

$$\left(L_{a_1} \,|\, L_{a_2} \;\; L_{a_3} \,|\, L_{a_4} \;\; L_{a_5} \,|\, L_{a_6} \,|\, L_{a_7} \right) \,|$$

$$\left. L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 16 \right\} \subseteq V,$$

$$H_3 = \left\{ \left(L_{a_1} \,|\, L_{a_2} \;\; 0 \,|\, 0 \;\; 0 \,|\, 0 \,|\, L_{a_3} \right), \left[\begin{array}{c|c|c|cc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \end{array} \right] \right|$$

$$\left. L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 20 \right\} \subseteq V,$$

$$H_4 = \left\{ \left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} \\ L_{a_{15}} & L_{a_{16}} \\ \hline L_{a_{17}} & L_{a_{18}} \end{array} \right], \left(L_{a_1} \,|\, L_{a_2} \;\; L_{a_3} \,|\, L_{a_4} \;\; L_{a_5} \,|\, 0 \,|\, L_{a_7} \right) \,\middle|\, \begin{array}{c} L_{a_i} \in L_{R^+ \cup \{0\}}; \\ 1 \le i \le 18 \end{array} \right\}$$

and

$H_5 = \left\{\left(L_{a_1} \mid L_{a_2}\ \ L_{a_3} \mid L_{a_4}\ \ L_{a_5} \mid L_{a_6} \mid L_{a_7}\right) \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 7\right\} \subseteq$ V be group super matrix semivector subspaces of refined labels of V over the group L_{Q^+}.

We see $= H_i \cap H_j = \phi$ if $i \neq j$, $1 \leq i, j \leq 5$. Further V = $\bigcup_{i=1}^{5} H_i$; thus V is the pseudo direct union of group super matrix semivector subspaces of refined labels over the group $L_{Q^+} = G$.

We can define group super matrix semivector spaces V and W of refined labels linear transformations provided V and W are defined over the same group G. We can also define subgroup super matrix semivector subspace of refined labels of V over the subgroup of G.

We will illustrate this situation by some examples.

Example 5.76: Let

$$V = \left\{ \left[\begin{array}{c|c} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \end{array}\right], \left(L_{a_1} \mid L_{a_2} \mid L_{a_3} \mid L_{a_4}\ \ L_{a_5}\ \ L_{a_6} \mid L_{a_7}\ \ L_{a_8} \mid L_{a_9}\right), \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ \hline L_{a_6} \\ L_{a_7} \\ L_{a_8} \\ \hline L_{a_9} \\ L_{a_{10}} \\ \hline L_{a_{11}} \end{array}\right], \right.$$

$$\left.\left[\begin{array}{c|ccc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ \hline L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ \hline L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \end{array}\right] \right| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 11 \right\}$$

be a group super row semivector space of refined labels over the group G = L_{R^+} under multiplication.

Consider

$$W = \left\{ \left[\begin{array}{c|c} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \end{array}\right], \left(L_{a_1} \,|\, L_{a_2} \,|\, L_{a_3} \,|\, L_{a_4} \;\; L_{a_5} \;\; L_{a_6} \,|\, L_{a_7} \;\; L_{a_8} \,|\, L_{a_9}\right) \right|$$

$L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 9 \Big\} \subseteq$ V; consider H = $L_{Q^+} \subseteq L_{R^+}$ = G, L_{Q^+} = H is a subgroup of G under multiplication.

We see W is a group super matrix semivector space of refined labels over the group H, thus W is a subgroup super matrix semivector subspace of refined labels over the subgroup H of the group G.

If G has no proper subgroups then we define V to be a pseudo simple group vector space of refined labels over G.

Likewise if V the group super matrix semivector space of refined labels over the group G has a subspace S which is over a semigroup H $\subseteq$ G we call S a pseudo semigroup super matrix semivector subspace of V over the semigroup H of the group G.

We will just illustrate this situation by some examples.

Example 5.77: Let

$$V = \left\{ \begin{bmatrix} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ \hline L_{a_8} \\ L_{a_9} \\ \hline L_{a_{10}} \end{bmatrix}, \left(L_{a_1} \mid L_{a_2} \quad L_{a_3} \mid L_{a_4} \quad L_{a_5} \mid L_{a_6} \quad L_{a_7} \mid L_{a_8} \right), \right.$$

$$\left. \left[\begin{array}{cc|c|ccc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_{13}} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{14}} \\ L_{a_{15}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \\ \hline L_{a_{22}} & L_{a_{23}} & L_{a_{24}} & L_{a_{25}} & L_{a_{26}} & L_{a_{27}} & L_{a_{28}} \\ L_{a_{29}} & L_{a_{30}} & L_{a_{31}} & L_{a_{32}} & L_{a_{33}} & L_{a_{34}} & L_{a_{35}} \end{array} \right] \middle| \; L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 35 \right\}$$

be a group super matrix semivector space of refined labels over the group G = L_{Q^+} under multiplication.

Take

$$P = \left\{ \left(L_{a_1} \mid L_{a_2} \quad L_{a_3} \mid L_{a_4} \quad L_{a_5} \mid L_{a_6} \quad L_{a_7} \mid L_{a_8} \right) \right.$$

$$\left.\left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ \hline L_{a_8} \\ L_{a_9} \\ \hline L_{a_{10}} \end{array}\right] \right| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 10 \Bigg\} \subseteq V$$

be a pseudo semigroup super matrix semivector subspace of refined labels of V over the semigroup

$$S = \left\{ L_{1/2^n} \,\middle|\, n = 0, 1, 2, \ldots, \infty \right\} \subseteq L_{Q^+}$$

under multiplication.

It is interesting to note that this V has infinitely many pseudo semigroup super matrix semivector subspaces of refined labels.

The following theorem guarantees the existence of pseudo semigroup super matrix semivector subspaces of refined labels.

Recall a group G is said to be a Smarandache definite group if it has a proper subset $H \subset G$ such that H is the semigroup under the operations of G. We see L_{Q^+} and L_{R^+} are Smarandache definite groups as they have infinite number of semigroups under multiplications.

THEOREM 5.1: *Let V be a group super matrix semivector space of refined labels over the group G =* L_{Q^+} *(or* L_{R^+} *). V has infinitely many pseudo semigroup super matrix semivector*

subspaces of refined labels of V over subsemigroups H under multiplication of the group G = L_{Q^+} *(or* L_{R^+} *).*

Interested reader can derive several related results. However we can define pseudo set super matrix semivector subspace of refined labels of a group super matrix semivector space of refined labels over the multiplicative group L_{Q^+} and L_{R^+}.

We will only illustrate this situation by an example or two.

Example 5.78: Let

$$V = \left\{ \left[\begin{array}{c|cc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \end{array}\right], \right.$$

$$\left(\begin{array}{c|cc|c|c|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \\ L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \end{array}\right)$$

$$\left. \left[\begin{array}{c|cc|ccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right] \middle| L_{a_i} \in L_{R^+\cup\{0\}}; 1 \le i \le 16 \right\}$$

be a group super matrix semivector space of refined labels over the multiplicative group G = L_{Q^+}.

Consider

$$P = \left\{ L_{1/2^n} \cup L_{1/5^n} \middle| n = 0, 1, 2, \ldots \right\} \subseteq L_{Q^+};$$

P is only a subset of $L_{Q^+} = G$.

Take

$$M = \left\{ \left[\begin{array}{c|cc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \end{array}\right], \left(\begin{array}{c|cc|c|c|cc|c} L_{a_1} & 0 & 0 & L_{a_4} & 0 & L_{a_6} & L_{a_7} & 0 \\ L_{a_9} & 0 & 0 & L_{a_{12}} & 0 & L_{a_{14}} & L_{a_{15}} & 0 \\ L_{a_1} & 0 & 0 & L_{a_4} & 0 & 0 & 0 & 0 \end{array}\right) \right|$$

$$L_{a_i} \in L_{R^+ \cup \{0\}}; i = 1, 2, \ldots, 9, 12, 14, 15 \Big\} \subseteq V.$$

M is a set super matrix semivector space of refined labels over the set $P \subseteq G$. Thus M is a pseudo set super matrix semivector space of refined labels of V over the set P of the multiplicative group G.

This pseudo set super matrix semivector space of refined labels can also be defined in case of semigroup super matrix semivector space of refined labels over the semigroup $L_{Q^+ \cup \{0\}}$ or $L_{R^+ \cup \{0\}}$.

The task of studying this notion and giving examples to this effect is left as an exercise to the reader.

Chapter Six

APPLICATION OF ALGEBRAIC STRUCTURES USING SUPER MATRICES OF REFINED LABELS

Applications of super matrix vector spaces of refined labels and super matrix semivector space of refined labels is at a very dormant state, as only in this book such concepts have been defined and described. The study of refined labels is very recent. The introduction of super matrix vector spaces of refined labels (ordinary labels) are very new. Certainly these new notions will find applications in the super fuzzy models, qualitative belief function models and other places were super matrix model can be adopted.

Since DSm are applied in several fields the interested researcher can find application in appropriate models where DSm finds its applications. When one needs the bulk work to save time super matrix of refined labels can be used for information retrieval, fusion and management.

Further as the study is very new the applications of these structures will be developed by researches in due course of time.

Chapter Seven

SUGGESTED PROBLEMS

In this chapter we have suggested over hundred problems some of which are open research problems, some of the problems are simple, mainly given to the reader for the better understanding of the definitions and results about the algebraic structure of the refined labels. We suggest problems which are both innovative and interesting.

1. Obtain some interesting properties about DSm super row vector space of refined labels over L_Q.

2. Obtain some interesting properties about DSm super column vector space of refined labels over L_R.

3. Find differences between the DSm super vector spaces of refined labels defined over L_R and L_Q.

4. Let V be a super row vector space given by;

$$V = \left\{ \left[\begin{array}{c|cc|c} L_{a_1} & L_{a_4} & L_{a_7} & L_{a_{10}} \\ L_{a_2} & L_{a_5} & L_{a_8} & L_{a_{11}} \\ L_{a_3} & L_{a_6} & L_{a_9} & L_{a_{12}} \end{array} \right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 12 \right\};$$

   a) Find a basis of V over L_R.
   b) What is the dimension of V over L_R?
   c) Give a subset of V which is linearly dependent.

5. Let $V = \left\{ \left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \mid L_{a_4} \quad L_{a_5} \right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 5 \right\}$ be a super row vector space of refined labels over the field L_R.
   a) Find a non invertible linear operator on V.
   b) Find an invertible linear operator on V.
   c) Find the algebraic structure enjoyed by $L_{L_R}(V, V)$.
   d) Does T : V $\rightarrow$ V be such that ker T is a hyper subspace of V?

6. Let $V = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \end{array} \right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 8 \right\}$ be a super column vector space of refined labels over L_R.

   Study questions (a) to (d) described in problem (5).

7. Obtain some interesting properties about super row matrix vector spaces of refined labels.

Let V = $\left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \\ L_{a_9} \\ L_{a_{10}} \\ \hline L_{a_{11}} \end{array} \right] \middle| \, L_{a_i} \in L_R; 1 \leq i \leq 11 \right\}$ be a super column vector space of refined labels over L_R.

a) Write V as a direct sum of super column vector subspaces of V over L_R.
b) Write V as a pseudo direct sum of super column vector subspaces of V over L_R.
c) Define a linear operator on V which is a projection.
d) Find dimension of V over L_R.
e) Find a basis of V over L_R.

8. Let

$$W = \left\{ \left(\begin{array}{cc|c|cc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ \hline L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \\ L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} & L_{a_{25}} \\ L_{a_{26}} & L_{a_{27}} & L_{a_{28}} & L_{a_{29}} & L_{a_{30}} \\ \hline L_{a_{31}} & L_{a_{32}} & L_{a_{33}} & L_{a_{34}} & L_{a_{35}} \end{array} \right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 35 \right\}$$

be a super matrix vector space of refined labels over the field L_R.

a) Find subspaces W_i in V so that $V = \bigcup_{i=1}^{5} W_i$ is a direct sum.
b) Find a basis for V over L_R.
c) Find dimension of V over L_R.

d) Suppose M =

$$\left\{ \left(\begin{array}{cc|c|cc} 0 & 0 & L_{a_1} & 0 & 0 \\ 0 & 0 & L_{a_2} & 0 & 0 \\ \hline L_{a_3} & L_{a_4} & 0 & L_{a_{12}} & 0 \\ L_{a_5} & L_{a_6} & 0 & 0 & L_{a_{13}} \\ L_{a_7} & L_{a_8} & 0 & L_{a_{14}} & 0 \\ L_{a_9} & L_{a_{10}} & 0 & 0 & L_{a_{15}} \\ \hline 0 & 0 & L_{a_{11}} & 0 & 0 \end{array} \right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 15 \right\} \subseteq$$

W be a super matrix vector subspace of V over L_R find $T : W \to W$ such that $T(W) \subseteq W$.
e) Is T invertible?
f) Find a non invertible T on W.
g) Write V as a pseudo direct sum.

9. Let P = $\left\{\left(\begin{array}{c|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array}\right) \middle| L_{a_i} \in L_R; 1 \le i \le 16\right\}$ be a super matrix vector space of refined labels over L_R.

a) Find a basis for P over L_R.
b) If L_R is replaced by L_Q what will be the dimension of P?
c) Write P as a pseudo direct sum of super vector subspaces of refined labels over L_R.

10. Let M = $\left\{\left(\begin{array}{ccc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ \hline L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \\ \hline L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \end{array}\right) \middle| L_{a_i} \in L_R; 1 \le i \le 20\right\}$ be a super matrix vector space over the refined labeled field L_R.

a) Prove M is infinite dimensional over L_Q.
b) Write M as a direct union of super matrix vector spaces.
c) Find S = $L_{L_R}(M, M)$; what is the algebraic structure enjoyed by S.

11. Let V = $\left\{\left(\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \end{array}\right) \middle| L_{a_i} \in L_R; 1 \le i \le 9\right\}$ be a super matrix vector space over L_R of refined labels.

$$M = \left\{ \left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} \end{array}\right] \middle| L_{a_i} \in L_R ; 1 \leq i \leq 14 \right\}$$ be a super matrix vector space of refined labels over L_R.

a) Define a $T : V \to M$ so that ker T is a nontrivial subspace of V.

b) Define $p : V \to M$ such that ker $p = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$.

c) Find the algebraic structure enjoyed by $S = \text{Hom}_{L_R}(V, M) = L_{L_R}(V, M)$.

d) What is the dimension of M over L_R?

e) Find $\eta : M \to V$ so that ker $\eta \neq \left[\begin{array}{cc} 0 & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array}\right]$.

f) Find $B = \text{Hom}_{L_R}(M, V) = L_{L_R}(M, V)$.

g) Compare B and S.

12. Let

$$M = \left\{ \left(\begin{array}{ccc|cc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ \hline L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ \hline L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \\ L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} & L_{a_{25}} \\ L_{a_{26}} & L_{a_{27}} & L_{a_{28}} & L_{a_{29}} & L_{a_{30}} \end{array} \right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 30 \right\}$$

be a super matrix vector space over L_R.

a) Let

$$P = \left\{ \left(\begin{array}{ccc|cc} 0 & 0 & 0 & L_{a_1} & L_{a_2} \\ \hline 0 & 0 & 0 & 0 & 0 \\ L_{a_3} & L_{a_4} & L_{a_5} & 0 & 0 \\ \hline 0 & 0 & 0 & L_{a_6} & L_{a_7} \\ 0 & 0 & 0 & L_{a_8} & L_{a_9} \\ 0 & 0 & 0 & L_{a_{10}} & L_{a_{11}} \end{array} \right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 11 \right\} \subseteq$$

M. Find $T : M \to M$ so that $T(P) \subseteq P$.
find $\eta : M \to M$ such that $\eta(P) \not\subseteq P$.

b) Let $V = \left\{ \left[\begin{array}{c|c|cc} L_{a_1} & L_{a_3} & L_{a_5} & L_{a_7} \\ L_{a_2} & L_{a_4} & L_{a_6} & L_{a_8} \end{array} \right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 8 \right\}$ be a super row vector, vector space of refined labels over L_R.

i) Define a linear function $f : V \to L_R$.
ii) Is ker f a super hyper space?

13. Obtain some interesting results on super matrix vector space of refined labels over L_R.

14. Let $V = L_R$ be a vector space (linear algebra) over L_Q. Prove dimension of V is infinite over L_Q.

15. Prove V = L_R is finite dimensional over L_R.

16. Let V = $\left\{\left[\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \end{array}\right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 9\right\}$ be a super matrix vector space of refined labels over L_R. Prove V is not a super matrix linear algebra of refined labels over L_R.

17. Prove a super matrix vector space of refined labels over L_R can never be a super matrix linear algebra of refined labels over L_R.

18. Let M = $\left\{\left(L_{a_1} \mid L_{a_2} \mid L_{a_3} \mid L_{a_4} \quad L_{a_5}\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 5\right\}$ be a super row vector space of refined labels over L_R. How many such super row vector spaces of refined labels can be constructed using 1 × 5 super row vectors by varying the partition on $\left(L_{a_1} \quad L_{a_2} \quad L_{a_3} \quad L_{a_4} \quad L_{a_5}\right)$?

19. Prove those classical theorems which can be adopted on super matrix vector space of refined labels.

20. Let

$$V = \left\{\left[\begin{array}{ccc|cc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_{17}} \\ L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} & L_{a_{18}} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{19}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} & L_{a_{20}} \end{array}\right] \middle| L_{a_i} \in L_{R^+\cup\{0\}}; 1 \leq i \leq 20\right\}$$

21. be a super row vector semivector space of refined labels over $L_{R^+\cup\{0\}}$.

    a) Find a basis for V over $L_{R^+\cup\{0\}}$.

b) Find a set of linearly independent elements of V which is not a basis of V over $L_{R^+ \cup \{0\}}$.

c) Find a linearly dependent subset of V over the semifield $L_{R^+ \cup \{0\}}$.

d) Prove V has linearly independent subsets whose cardinality is greater than that of the basis.

21. Let $V = \left\{ \left[\begin{array}{cccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ \hline L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \\ L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \\ \hline L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ L_{a_{25}} & L_{a_{26}} & L_{a_{27}} & L_{a_{28}} \\ L_{a_{29}} & L_{a_{30}} & L_{a_{31}} & L_{a_{32}} \end{array} \right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 32 \right\}$

be super column vector, semivector space of refined labels over the semifield $L_{R^+ \cup \{0\}}$.

a) Find two super column vector semivector subspaces of V which has zero intersection over $L_{R^+ \cup \{0\}}$.

b) Write V as a direct sum of super column vector semivector subspaces over $L_{R^+ \cup \{0\}}$.

c) Write V as a pseudo direct sum of super column vector semivector subspaces over $L_{R^+ \cup \{0\}}$.

d) Find a $T : V \to W$ which has a non trivial kernel.

e) Find a $\eta : V \to V$ so that the η is an invertible linear transformation on V.

f) Can $\eta : V \to V$ be one to one yet η^{-1} cannot exist? Justify.

22. Let V = $\left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ \hline L_{a_8} \\ L_{a_9} \\ L_{a_{10}} \\ \hline L_{a_{11}} \\ L_{a_{12}} \\ \hline L_{a_{13}} \end{array} \right] \middle| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \le i \le 13 \right\}$ be a super column semivector space of refined labels over $L_{Q^+ \cup \{0\}}$.

a) Find a basis of V over $L_{Q^+ \cup \{0\}}$.
b) Can V have more than one basis?
c) Find dimension of V over $L_{Q^+ \cup \{0\}}$.
d) Write V as a pseudo direct sum of super column semivector subspaces over $L_{Q^+ \cup \{0\}}$.

e) Suppose W = $\left\{ \left[\begin{array}{c} L_{a_1} \\ \hline 0 \\ L_{a_2} \\ \hline 0 \\ L_{a_3} \\ 0 \\ L_{a_4} \\ \hline 0 \\ L_{a_5} \\ 0 \\ \hline L_{a_6} \\ 0 \\ \hline L_{a_7} \end{array} \right] \middle| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \le i \le 7 \right\} \subseteq V$ be a super column semivector subspace of V of refined labels over $L_{Q^+ \cup \{0\}}$.

i) Find a $T : V \to V$ such that $T(W) \subseteq W$.

ii) Find $T_1 : V \to V$ such that $T_1(W) \not\subseteq W$.

f) Find a subset of V with more than 14 elements which is a linearly independent subset of V over $L_{Q^+ \cup \{0\}}$.

23. Let V = $\left\{ \left[\begin{array}{cccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ \hline L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \\ L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \\ \hline L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \end{array} \right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 24 \right\}$

be a super matrix semivector space of refined labels over $L_{Q^+ \cup \{0\}}$.

a) Is V finite dimensional over $L_{Q^+ \cup \{0\}}$?
b) Can V have finite linearly independent subset?
c) Can V have finite linearly dependent subsets?
d) Find $\text{Hom}_{L_{Q^+ \cup \{0\}}}(V, V) = S$.
e) What is the algebraic structure enjoyed by S?
f) Is S a finite set?
g) Give two maps $T : V \to V$ and $P : V \to V$ so that TP = PT.
h) Find $T_f : V \to L_{Q^+ \cup \{0\}}$.
i) If $K = \{f \mid f : V \to L_{Q^+ \cup \{0\}}\}$, what is the cardinality of K?
j) Does K have any nice algebraic structure on it?

24. Let

$$V = \left\{ \left[\begin{array}{c|c|c|cc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ \hline L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ \hline L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \end{array} \right] \middle| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \le i \le 20 \right\}$$

be a super matrix semivector space of refined labels over $L_{Q^+ \cup \{0\}}$} and

$$W = \left\{ \left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ \hline L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ \hline L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \\ L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ L_{a_{25}} & L_{a_{26}} & L_{a_{27}} \end{array} \right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 27 \right\}$$

be a super column vector semivector space of refined labels over $L_{Q^+ \cup \{0\}}$.

a) Define T : V → W such that T is an invertible linear transformation on V.
b) Prove W is an infinite dimensional space.
c) If $\eta : V \to W$ does η enjoy any special property related with dimensions of V and W?
d) Can P : V → W such that ker P is different from the zero space?
e) What is the algebraic structure enjoyed by $\text{Hom}_{L_{Q^+ \cup \{0\}}}(V, W)$?

25. Let V be a super row vector space of refined labels over L_R; where $V = \left\{ \left(L_{a_1} \quad L_{a_2} \mid L_{a_3} \mid \ldots \mid L_{a_n} \right) \middle| L_{a_i} \in L_R; 1 \le i \le n \right\}$;

a) Find $\text{Hom}_{L_R}(V, V)$.
b) Is $\text{Hom}_{L_R}(V, V)$ a super row vector space?
c) Is $\text{Hom}_{L_R}(V, V)$ just a vector space of refined labels? Justify your claim.

26. Let

$$V = \left\{ \left[\begin{array}{cc|ccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ \hline L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \\ \hline L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} & L_{a_{25}} \end{array} \right] \middle| L_{a_i} \in L_Q; 1 \le i \le 25 \right\}$$

be a super matrix vector space of refined labels over the refined field L_Q.

a) What is the dimension of V over L_Q?
b) Does V have super matrix row vector subspace over L_Q?
c) Find a set linearly dependent elements in V over L_Q.
d) Find the algebraic structure enjoyed by $L_{L_Q}(V, V) = \text{Hom}_{L_Q}(V, V)$.

27. Let V = $\left\{ \left[\begin{array}{cccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \\ L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \\ L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ \hline L_{a_{25}} & L_{a_{26}} & L_{a_{27}} & L_{a_{28}} \\ L_{a_{29}} & L_{a_{30}} & L_{a_{31}} & L_{a_{32}} \\ L_{a_{33}} & L_{a_{34}} & L_{a_{35}} & L_{a_{36}} \end{array}\right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 36 \right\}$ be a super matrix vector space of refined labels over L_R.

a) Find a super matrix vector subspace W of V dimension 19 of refined labels over L_R.
b) Find T : V $\to$ V so that T (W) $\subseteq$ W.
c) Find P : V $\to$ V so that P (W) $\not\subseteq$ W.
d) Compare the super linear operators P and T of V.
e) What is ker T?
f) Find ker P and compare it with ker T.
g) Find a super matrix vector subspace of dim 5 of V of refined labels over L_R.
h) Write V as a pseudo direct sum of super matrix vector subspaces.

28. Let

$$V = \left\{ \left[\begin{array}{c|cc|ccc} L_{a_1} & L_{a_5} & L_{a_9} & L_{a_{13}} & L_{a_{17}} & L_{a_{21}} \\ L_{a_2} & L_{a_6} & L_{a_{10}} & L_{a_{14}} & L_{a_{18}} & L_{a_{22}} \\ L_{a_3} & L_{a_7} & L_{a_{11}} & L_{a_{15}} & L_{a_{19}} & L_{a_{23}} \\ L_{a_4} & L_{a_8} & L_{a_{12}} & L_{a_{16}} & L_{a_{20}} & L_{a_{24}} \end{array}\right] \middle| \begin{array}{l} L_{a_i} \in L_R; \\ 1 \leq i \leq 24 \end{array} \right\}$$

be a super matrix vector space of refined labels over L_Q.

a) Find dimension of V over L_Q.
b) Find $\mathrm{Hom}_{L_Q}(V, V)$.

c) Write V = $\bigcup_{i=1}^{6} W_i$ so that V is a direct union of super matrix vector subspaces of V over L_Q.

d) Write V = $\bigcup_{i=1}^{6} W_i$ so that V is the pseudo direct union of super matrix vector subspaces of V.

29. Let M = $\left\{\left[\begin{array}{c|cc|cc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_2} & L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_3} & L_{a_7} & L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_8} & L_{a_9} & L_{a_4} & L_{a_5} \\ L_{a_5} & L_{a_6} & L_{a_3} & L_{a_5} & L_{a_6} \end{array}\right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 9\right\}$

be a super matrix vector space of refined labels over L_R.

a) Find dimension of V over L_R.
b) Is dim V over R less than 25?
c) Can V have a super matrix vector subspaces of refined labels of dimension 9 over L_R?
d) Find the dimension of the super matrix vector subspace

$$P = \left\{\left[\begin{array}{c|cc|cc} L_{a_1} & 0 & 0 & 0 & 0 \\ \hline 0 & L_{a_6} & L_{a_7} & 0 & 0 \\ 0 & L_{a_7} & L_{a_1} & 0 & 0 \\ \hline 0 & 0 & 0 & L_{a_4} & L_{a_5} \\ 0 & 0 & 0 & L_{a_5} & L_{a_6} \end{array}\right] \middle| L_{a_1}, L_{a_7}, L_{a_4}, L_{a_5} \in L_R\right\} \subseteq V$$

e) Find a linear operator T on V such that T (P) $\subseteq$ P.
f) Let M =

$$\left\{\left[\begin{array}{c|cc|cc} 0 & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ \hline L_{a_2} & 0 & 0 & L_{a_8} & L_{a_9} \\ L_{a_3} & 0 & 0 & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & 0 & 0 & 0 & 0 \\ L_{a_5} & 0 & 0 & 0 & 0 \end{array}\right] \middle| L_{a_2}, L_{a_3}, L_{a_4}, L_{a_5}, L_{a_8}, L_{a_9} \in L_R\right\}$$

$\subseteq$ V, be a super matrix vector space of refined labels over L_R of V.

Find dimension of M over L_R.

g) Find T : V $\to$ V so T (M) $\not\subseteq$ M.

30. Let V = $$\left\{\left.\left[\begin{array}{c|ccc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ \hline L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ \hline L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ \hline L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \end{array}\right] \right| L_{a_i} \in L_R; 1 \leq i \leq 5\right\}$$

be a super matrix vector space of refined labels over L_R.

a) What is dimension of V over L_R?

b) Is V of dimension 45 over L_R? Justify your answer.

c) Define T : V $\to$ V so that ker T = $$\left[\begin{array}{c|ccc|c} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

d) Find P_1 and P_2 two distinct linear operators of V so that $P_1P_2 = P_2 P_1$ = Identity operator on V.

31. Give some interesting applications of super matrix vector spaces of refined labels over L_R or L_Q.

32. Prove we cannot in general construct super matrix linear algebras of refined labels defined over L_R or L_Q.

33. Obtain some interesting properties enjoyed by integer set super matrix semivector space of refined labels defined over the set $S = 3Z^+ \cup 5Z^+ \cup \{0\}$.

34. Obtain some interesting properties by group super matrix semivector spaces of refined labels defined over L_{Q^+} or L_{R^+}.

35. Let V =

$$\left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \\ L_{a_9} \\ L_{a_{10}} \\ \hline L_{a_{11}} \end{array}\right] \left[\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \end{array}\right] \left[\begin{array}{c|cc|ccc|cc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array}\right] \right|$$

$L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 18 \}$ be a integer super matrix semivector space of refined labels over $3Z^+ \cup 5Z^+ \cup \{0\} = S$.

a) Find a basis of V over S.
b) Find integer super matrix semivector subspace of V of refined labels.
c) Write $V = \bigcup_{i=1}^{3} W_i$ as a direct sum.

36. Let V =

$$\left\{\left[\begin{array}{c|cc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \end{array}\right], \left[\begin{array}{c|c|c} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \end{array}\right], \left[\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \end{array}\right] \middle| L_{a_i} \in L_Q; 1 \le i \le 9\right\}$$

be a set super matrix vector space of refined labels over the set S = $L_{Q^+ \cup \{0\}} \cup L_{R^+ \cup \{0\}}$.

a) Find subspaces of V.
b) Write V as a pseudo direct sum of subspaces.

37. Let V =

$$\left\{\left(\begin{array}{c|c} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \end{array}\right), \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ \hline L_{a_3} \\ L_{a_4} \\ \hline L_{a_5} \\ L_{a_6} \\ L_{a_7} \\ \hline L_{a_8} \\ \hline L_{a_9} \end{array}\right], \left[\begin{array}{c|ccc|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} \\ L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} \end{array}\right] \middle| \right.$$

$L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 9\}$ be a group super matrix semivector space of refined labels over L_{R^+} the multiplicative group.

a) Find V= $\bigcup_{i=1} W_i$ as a pseudo direct sum.
b) Find subgroup super matrix semivector subspace of V of refined labels over L_{R^+}.
c) Find pseudo semigroup super matrix semivector subspaces of V of refined labels over L_{R^+}.

d) Find pseudo set super matrix semivector subspace of V of refined labels over L_{R^+}.

38. Let V =

$$\left\{\left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ L_{a_3} & L_{a_4} \\ L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} \\ L_{a_5} & L_{a_6} \\ L_{a_5} & L_{a_6} \\ L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} \end{array}\right], \left[\begin{array}{c|c} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} \end{array}\right], \left[\begin{array}{c|cc|ccc|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} \end{array}\right] \right|$$

$L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 9\}$ be a semigroup super matrix semivector space of refined labels over the semigroup S = $L_{Q^+ \cup \{0\}}$ under addition.

a) Write V = $\bigcup W_i$ as a pseudo direct sum of subspaces.

b) Find two pseudo set super matrix semivector subspaces of refined labels of V over S.

c) Find T : V $\rightarrow$ V with non trivial kernel.

d) Find P : V $\rightarrow$ V so that P^{-1} exists.

e) If K = $\left\{\left[\begin{array}{c|c} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} \end{array}\right] \middle| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \le i \le 2\right\} \subseteq$ V; find dimension of K over $L_{Q^+ \cup \{0\}}$.

39. Obtain some nice applications of semigroup super matrix semivector space of refined labels over the semigroup $L_{R^+ \cup \{0\}}$ under addition.

40. Let $V = \left\{ \left[\begin{array}{c|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \\ \hline L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \\ L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ \hline L_{a_{25}} & L_{a_{26}} & L_{a_{27}} & L_{a_{28}} \end{array}\right] \middle| L_{a_i} \in L_R ; 1 \leq i \leq 28 \right\}$ be a super matrix vector space of refined labels over L_R.

a) Find dimension of V over L_R.
b) Find subspaces of dimensions 5, 7 and 20.
c) Write $V = W_1 \oplus \ldots \oplus W_t$ where W_i 's are subspaces of V.
d) Find $T : V \to V$ so that T is an idempotent operator on V.
e) Find $T : V \to V$ so that T^{-1} exists.

41. Let

$$P = \left\{ \left[\begin{array}{c|c} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \end{array}\right], \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ \hline L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \\ \hline L_{a_7} \\ L_{a_8} \\ \hline L_{a_9} \end{array}\right], \left[\begin{array}{c|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \\ \hline L_{a_{17}} & L_{a_{18}} & L_{a_{20}} & L_{a_{19}} \end{array}\right] \middle| \begin{array}{l} L_{a_i} \in L_R ; \\ 1 \leq i \leq 19 \end{array} \right\}$$

be a set super matrix vector space of refined labels over the set

$$S = \left\{ L_{\frac{1}{2^n} \cup \{0\}} \cup L_{\frac{1}{5^n} \cup \{0\}} \;\middle|\; n \in 0, 1, 2, \ldots, \infty \right\} \subseteq L_R.$$

a) Write $V = \bigcup_{i=1}^{3} W_i$ as a direct sum of subspaces.

b) Write $V = \bigcup_{i=1}^{5} W_i$ as a pseudo direct sum of subspaces?

c) Find two disjoint subspaces W_1 and W_2 and find two linear operators T_1 and T_2 so that $T_1 (W_1) \subseteq W_1$ and $T_2 (W_2) \subseteq W_2$.

d) Find a subset super matrix vector subspace of refined labels of V.

42. Write down any special feature enjoyed by the super matrix semivector space of refined labels over $L_{R^+ \cup \{0\}}$ or $L_{Q^+ \cup \{0\}}$.

43. Can we have finite super matrix semivector space of refined labels over the semifield $L_{R^+ \cup \{0\}}$ or $L_{Q^+ \cup \{0\}}$? Justify your claim.

44. Is it possible to construct set super matrix semivector space of refined labels over $L_{Q^+ \cup \{0\}}$ of finite order?

45. Let $P = \left\{ \left[\begin{array}{c|ccc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ \hline L_{a_2} & L_{a_1} & L_{a_5} & L_{a_3} & L_{a_4} \\ L_{a_3} & L_{a_5} & L_{a_4} & L_{a_1} & L_{a_2} \\ L_{a_4} & L_{a_3} & L_{a_1} & L_{a_2} & L_{a_5} \\ \hline L_{a_5} & L_{a_4} & L_{a_2} & L_{a_5} & L_{a_3} \end{array} \right] \;\middle|\; L_{a_i} \in L_R; 1 \leq i \leq 5 \right\}$ be a super matrix vector space of refined labels over the field L_R.

a) Find dimension of P over L_R.

b) Write $P = \cup W_i$, W_i, subspaces as a direct sum.

c) Can P have ten dimensional super matrix vector subspace of refined labels over L_R? Justify your answer.

d) Define $T : P \to P$ so that $T^2 = T$.

e) Find $S : P \to P$ so that $S^2 = (0)$.
f) Find hyper subspaces of refined labels in P over L_R.

46. Obtain some interesting properties about $L_{L_R}(V, L_R)$ where V is a super matrix vector space of refined labels over L_R.

47. Let V = $$\left\{ \left[\begin{array}{c|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ \hline L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \end{array} \right] \middle| L_{a_i} \in L_R; 1 \le i \le 8 \right\}$$ be a super matrix vector space of refined labels over L_R. Define $f : V \to L_R$ so that f is a linear functional on V. Suppose $v_1, \ldots, v_6$ are some six linearly independent elements of V; find $f(v_i)$; $i = 1, \ldots, 6$. Study the set $\{f(v_1), \ldots, f(v_6)\} \subseteq L_R$. What is the algebraic structure enjoyed by $L_{L_R}(V, L_R)$? How many hyper subspaces exist in case of V?

48. Let V = $$\left\{ \left[\begin{array}{c|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{a_2} & L_{a_3} & L_{a_4} & L_{a_1} \\ L_{a_3} & L_{a_4} & L_{a_2} & L_{a_1} \\ \hline L_{a_4} & L_{a_1} & L_{a_3} & L_{a_3} \end{array} \right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 4 \right\}$$ be a integer super matrix semivector space of refined labels over $Z^+ \cup \{0\}$.
a) Find a linear operator T on V which is non invertible.
b) Write V as a direct sum of semivector subspaces of refined labels over $Z^+ \cup \{0\}$.

49. Let $M = \left\{ \left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ \hline L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \\ L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \\ L_{a_{25}} & L_{a_{26}} & L_{a_{27}} \\ L_{a_{28}} & L_{a_{29}} & L_{a_{30}} \end{array} \right] \middle| L_{a_i} \in L_{Q^+ \cup \{0\}}; 1 \le i \le 30 \right\}$ be a super matrix semivector space of refined labels over the semifield $L_{Q^+ \cup \{0\}}$.

a) Find the dimension of M over $L_{Q^+ \cup \{0\}}$.

b) Write M as a pseudo direct sum of semivector subspaces of refined labels over $L_{Q^+ \cup \{0\}}$.

c) Write $M = \bigcup_i W_i$ as a pseudo direct sum of super matrix semivector subspaces of refined labels over $L_{Q^+ \cup \{0\}}$.

d) Show in M we can have more number of linearly independent elements than the cardinality of the basis of M over $L_{Q^+ \cup \{0\}}$.

50. Let $V = \left\{ \left[\begin{array}{c|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{a_2} & 0 & L_{a_4} & L_{a_5} \\ 0 & 0 & L_{a_3} & 0 \\ \hline L_{a_7} & L_{a_5} & 0 & L_{a_9} \\ 0 & L_{a_{10}} & L_{a_1} & 0 \end{array} \right] \middle| L_{a_i} \in L_R; 1 \le i \le 11 \right\}$ be a super matrix vector space over L_R.

a) Find dimension of V over L_R.
b) Write $\cup W_i = V$ as a pseudo direct sum of subspaces.
c) Is the number of subspaces of V over L_R finite or infinite?
d) Give a linearly dependent subset of order 10.

51. Let V = $\left\{ \left[\begin{array}{c|ccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{a_2} & 0 & 0 & L_{a_5} \\ L_{a_3} & 0 & L_{a_6} & 0 \\ L_{a_4} & L_{a_7} & 0 & 0 \end{array} \right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 7 \right\}$ be a group super matrix semivector space of refined labels over multiplicative group G = L_{R^+}.

a) What is the dimension of V over G?
b) Prove there exist infinite number of pseudo set super matrix semivector subspaces of V over L_R.
c) Prove there exists infinite number of pseudo semigroup super matrix semivector subspaces of V over L_R.
d) Prove V is not a simple group super matrix semivector space over L_R.
e) Write V = $\bigcup_{i=1}^{7} W_i$ as direct sum of subspaces of V.
f) Write V=$\bigcup_{i=1}^{6} W_i$ as pseudo direct sum of subspaces of V

52. Let

$$V = \left\{ \left[\begin{array}{c|cc|cccc|c} L_{a_1} & L_{a_4} & L_{a_7} & L_{a_{10}} & L_{a_{13}} & L_{a_{16}} & L_{a_{19}} & L_{a_{22}} \\ L_{a_2} & L_{a_5} & L_{a_8} & L_{a_{11}} & L_{a_{14}} & L_{a_{17}} & L_{a_{20}} & L_{a_{23}} \\ L_{a_3} & L_{a_6} & L_{a_9} & L_{a_{12}} & L_{a_{15}} & L_{a_{18}} & L_{a_{21}} & L_{a_{24}} \end{array} \right] \middle| \begin{array}{l} L_{a_i} \in L_R; \\ 1 \le i \le 24 \end{array} \right\}$$

be a super row vector space over L_R of refined labels.
a) What is the dimension of V?
b) Find $\text{Hom}_{L_R}(V, V)$.
c) Find T : V $\to$ V so that $T^2 = T$.
d) Find T_1 : V $\to$ V so that $T_1^2 = (0)$.

53. Let $P = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ L_{a_6} \end{array}\right], \left[\begin{array}{c} L_{a_1} \\ L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ \hline L_{a_6} \end{array}\right], \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ \hline L_{a_3} \\ \hline L_{a_4} \\ \hline L_{a_5} \\ \hline L_{a_6} \end{array}\right], \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ \hline L_{a_6} \end{array}\right] \middle| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 6 \right\}$

be a set super matrix semivector space of refined labels over the set $S = L_{2^n \cup \{0\}} \cup L_{3^n \cup \{0\}} \cup L_{5^n}$.

a) Find subspaces of P.
b) What is dimension of P over S?
c) Express $P = \bigcup_{i=1}^{4} P_i$ as a direct sum of subspaces of V.
d) Define $T : P \to P$ with $T.T = T$.
e) Define $T_1 : P \to P$ so that T_1^{-1} exists.

54. Let

$$V = \left\{ \left[\begin{array}{c|ccc} 0 & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{a_2} & 0 & L_{a_3} & L_{a_4} \\ L_{a_3} & L_{a_3} & 0 & L_{a_2} \\ L_{a_4} & L_{a_4} & L_{a_2} & 0 \end{array}\right], \left[\begin{array}{c|c} L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} \end{array}\right], \left[\begin{array}{cc|cc} 0 & L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_1} & 0 & L_{a_4} & L_{a_5} \\ \hline L_{a_2} & L_{a_4} & 0 & L_{a_6} \\ L_{a_3} & L_{a_5} & L_{a_6} & 0 \end{array}\right] \right|$$

$L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 6 \}$ be a integer set super matrix semivector space of refined labels over the set of integers $S = 3Z^+ \cup \{0\} \cup 2Z^+$. Suppose

$W = \{ [L_{a_1} \;\; L_{a_2} \mid L_{a_3} \;\; L_{a_4} \;\; L_{a_5} \mid L_{a_6} \mid L_{a_7} \;\; L_{a_8} \;\; L_{a_9} \;\; L_{a_{10}}],$

$$\left.\left[\begin{array}{c|c|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \end{array}\right], \left[\begin{array}{c|c} L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \\ \hline L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \end{array}\right] \right| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 6 \Bigg\}$$

be an integer set super matrix semivector space of refined labels over the set $S = 3Z^+ \cup 2Z^+ \cup \{0\}$.

a) Find a linear transformation T from V into W so that T^{-1} exists.

b) Find the algebraic structure enjoyed by $Hom_S(V, W)$.

c) Find T_1 and T_2 two linear transformation from V into V such that $T_1 \cdot T_2$ is an idempotent linear transformation.

d) Find two linear transformation L_1 and L_2 from W into W so that $L_1 \cdot L_2 = L_1 \cdot L_2$ but $L_1 \cdot L_2 = L_1 \cdot L_2 \neq I$.

55. Let

$$M = \left\{ \left[\begin{array}{c|ccc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ \hline L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ \hline L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \\ \hline L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} & L_{a_{25}} \\ L_{a_{26}} & L_{a_{27}} & L_{a_{28}} & L_{a_{29}} & L_{a_{30}} \\ L_{a_{31}} & L_{a_{32}} & L_{a_{33}} & L_{a_{34}} & L_{a_{35}} \\ \hline L_{a_{36}} & L_{a_{37}} & L_{a_{38}} & L_{a_{39}} & L_{a_{40}} \\ L_{a_{41}} & L_{a_{42}} & L_{a_{43}} & L_{a_{44}} & L_{a_{45}} \\ \hline L_{a_{46}} & L_{a_{47}} & L_{a_{48}} & L_{a_{49}} & L_{a_{50}} \end{array}\right] \middle| \begin{array}{c} L_{a_i} \in L_{Q^+ \cup \{0\}}; \\ 1 \leq i \leq 50 \end{array} \right\}$$ be a

super matrix vector space of refined labels over $L_{Q^+ \cup \{0\}}$.

a) Write $M = \bigcup_i W_i$; W_i super matrix vector subspaces of refined labels over $L_{Q^+ \cup \{0\}}$ of M as a direct sum.

b) Let W = $\left\{ \left[\begin{array}{c|ccc|c} 0 & 0 & 0 & 0 & 0 \\ \hline L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ 0 & 0 & 0 & 0 & 0 \\ \hline L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right] \middle| \begin{array}{c} L_{a_i} \in L_{Q^+ \cup \{0\}}; \\ 1 \leq i \leq 20 \end{array} \right\} \subseteq$

M be a super matrix vector subspace of M of refined labels over $L_{Q^+ \cup \{0\}}$.

Find T : M → M so that T (W) ⊆ W.

c) What is dimension of W?

d) Find T_1, T_2 : M → M so that (T_1 . T_2) (W) ⊆ W.

56. Obtain some special properties enjoyed by set super matrix vector space of ordinary labels over a suitable set.

57. Let V = $\left\{ \left[L_{a_1} \mid L_{a_2} \quad L_{a_3} \mid L_{a_4} \quad L_{a_5} \right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 5 \right\}$ be a super row matrix vector space of refined labels over L_R.
Will V ≅ $L_R \times L_R \times L_R \times L_R \times L_R$?
Justify your answer.

58. How many super row matrices of dimension seven can be built using the row matrix
$\left(L_{a_1} \; L_{a_2} \; L_{a_3} \; L_{a_4} \; L_{a_5} \; L_{a_6} \; L_{a_7} \right)$ of refined labels?

59. Prove or disprove that if V is a super matrix vector space of dimension n of refined labels defined over L_R then V is isomorphic with all super matrix vector space of dimension n of refined labels defined over L_R.

60. Let V = $\left\{\left.\left[\begin{array}{c|c|c} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \end{array}\right]\right| L_{a_i} \in L_R; 1 \leq i \leq 15\right\}$ be a super matrix vector space of refined labels defined over L_R. W = $\left\{\left.\left[\begin{array}{c|ccc|c} L_{a_1} & L_{a_4} & L_{a_5} & L_{a_6} & L_{a_7} \\ L_{a_2} & L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} \\ L_{a_3} & L_{a_{12}} & L_{a_{13}} & L_{a_{15}} & L_{a_{14}} \end{array}\right]\right| L_{a_i} \in L_R; 1 \leq i \leq 15\right\}$ be a super matrix vector space of refined labels defined over L_R. Is $W \cong V$?

61. Let

$$V = \left\{\left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ \hline L_{a_4} \\ L_{a_5} \\ \hline L_{a_6} \\ L_{a_7} \end{array}\right], \left[\begin{array}{c|c} L_{a_1} & L_{a_2} \\ L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \end{array}\right], \left[\begin{array}{ccc|cc|c} L_{a_1} & L_{a_5} & L_{a_9} & L_{a_{13}} & L_{a_{17}} & L_{a_{21}} \\ L_{a_2} & L_{a_6} & L_{a_{10}} & L_{a_{14}} & L_{a_{18}} & L_{a_{22}} \\ L_{a_3} & L_{a_7} & L_{a_{11}} & L_{a_{15}} & L_{a_{19}} & L_{a_{23}} \\ L_{a_4} & L_{a_8} & L_{a_{12}} & L_{a_{16}} & L_{a_{20}} & L_{a_{24}} \end{array}\right] \right|$$

$$L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 24\Big\}$$

and

$$W = \left\{\left[\begin{array}{c|cc} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \end{array}\right], \left(L_{a_1} \,|\, L_{a_2} \;\; L_{a_3} \;\; L_{a_4} \,|\, L_{a_5} \;\; L_{a_6} \,|\, L_{a_7}\right),\right.$$

$$\left.\left[\begin{array}{c|cc|ccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ L_{a_{19}} & L_{a_{20}} & L_{a_{21}} & L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \end{array}\right] \right| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 24 \right\}$$

be a group super matrix semivector space of refined labels over the multiplicative group $G = L_{R^+}$.

a) Is $V \cong W$?
b) Find dimension V.
c) Find $T : V \to W$ so that T^{-1} exists.
d) Find $T_1 : V \to W$ so that $T_1 \,.\, T_1 = T_1$.
e) Find pseudo set super matrix semivector subspaces M and P of V and W respectively over the set $B = L_{1/2^n} \cup L_{1/5^n} \subseteq L_{R^+}$ such that $M \cong P$.
f) Define $f : V \to W$ so that $f(M) = P$.

62. Let

$$V = \left\{ \left[\begin{array}{c|cccc} L_{a_1} & L_{a_2} & L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_3} & L_{a_4} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_5} & L_{a_6} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \end{array}\right], \left[\begin{array}{cc|c} L_{a_1} & L_{a_2} & L_{a_3} \\ L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \end{array}\right], \right.$$

$$\left. \left[\begin{array}{ccc} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ \hline L_{a_{16}} & L_{a_{17}} & L_{a_{18}} \\ L_{a_{19}} & L_{a_{20}} & L_{a_{21}} \\ \hline L_{a_{22}} & L_{a_{23}} & L_{a_{24}} \end{array}\right], \left[\begin{array}{c|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array}\right] \right| L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \le i \le 24 \right\}$$

be a semigroup super matrix semivector space of refined labels over the semigroup $L_{R^+ \cup \{0\}}$ under addition.

a) Find dimension V.
b) Find T : V → V so that T . T = T.
c) Find P : V → V so that $P^2 = 0$.
d) Find M : V → V so that ker M is a semigroup super matrix semivector subspace of V of refined labels over the additive semigroup $L_{R^+ \cup \{0\}}$.
e) Find subsemigroup super matrix semivector subspace of refined labels of V.
f) Find pseudo set super matrix semivector subspaces of V of refined labels over subsets of the additive semigroup $L_{R^+ \cup \{0\}}$.

63. Give an example of a group super matrix semivector space of refined labels which is simple.

64. Obtain some interesting properties about group super square matrix semivector space of refined labels over the multiplicative group L_{Q^+}.

65. Let $V = \left\{\left(L_{a_1}\ L_{a_2}\ L_{a_3}\ L_{a_4}\ \middle|\ L_{a_5}\ L_{a_6}\ L_{a_7}\ \middle|\ L_{a_8}\ L_{a_9}\ \middle|\ L_{a_{10}}\right) \middle|\ L_{a_i} \in L_R; 1 \leq i \leq 10\right\}$ be a super row matrix vector space of refined labels over L_R.

a) Find for f : V → L_R, a linear functional defined by

$$f\left(L_{a_1}\ L_{a_2}\ L_{a_3}\ L_{a_4}\ \middle|\ L_{a_5}\ L_{a_6}\ L_{a_7}\ \middle|\ L_{a_8}\ L_{a_9}\ \middle|\ L_{a_{10}}\right)$$

$= L_{a_1+a_2+\ldots+a_{10}}$; the ker f.

b) Find $L_{L_R}(V, L_R)$.
c) What is dimension of $L_{L_R}(V, L_R)$?
d) What is the algebraic structure enjoyed by $L_{L_R}(V, L_R)$?
e) What is the dimension of V over L_R?
f) Find $\text{Hom}_{L_R}(V, V)$.
g) What is the algebraic structure enjoyed by $\text{Hom}_{L_R}(V, V)$?

66. Find some interesting properties enjoyed by $L_{L_R}(V, L_R)$ = $\{f : V \to L_R\}$ where V is a super matrix vector space of refined labels defined over L_R.

67. Find the algebraic structure enjoyed by $H_S(V,V)$ where V is a set super matrix semivector space of refined labels defined over the set S = $L_{R^+ \cup \{0\}}$.

68. Let V = $\left\{ \left[\begin{array}{ccc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array} \right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 16 \right\}$ be a super matrix vector space of refined labels over L_R.

$$W = \left\{ \left[\begin{array}{c|ccc} 0 & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{-a_2} & 0 & L_{a_5} & L_{a_6} \\ L_{-a_3} & L_{-a_5} & 0 & L_{-a_7} \\ L_{-a_4} & L_{-a_6} & L_{a_7} & 0 \end{array} \right] \middle| \begin{array}{c} L_{a_i} \in L_R; \\ 1 \leq i \leq 7; \\ L_{-a_i} = -L_{a_i} \end{array} \right\}$$

be a super matrix vector space of refined labels over L_R.

a) Is $V \cong W$?
b) Find dim W over L_R.
c) Find dim V over L_R.
d) Study $\text{Hom}_{L_R}(V, W)$ and $\text{Hom}_{L_R}(W, V)$.
e) Find $T : V \to W$ so that $T . T = T$.
f) Find $T_1 : V \to W$ so that T_1^{-1} exists.

69. Let $V = \left\{ \left[\begin{array}{cc} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} \\ L_{a_{15}} & L_{a_{16}} \\ \hline L_{a_{17}} & L_{a_{18}} \\ L_{a_{19}} & L_{a_{20}} \end{array}\right], \left[\begin{array}{c|ccc} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \\ L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \end{array}\right], \right.$

$\left[\begin{array}{c|ccc|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & 0 & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} \\ L_{a_{14}} & L_{a_{15}} & L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \end{array}\right],$

$\left. \left[\begin{array}{c|cc|cc} 0 & L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{a_5} & 0 & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_9} & L_{a_{10}} & 0 & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} & 0 & L_{a_{16}} \\ L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} & 0 \end{array}\right] \right| L_{a_i} \in L_R; 1 \le i \le 20 \right\}$

be a semigroup super matrix semivector space of refined labels over $L_{R^+ \cup \{0\}}$ under addition.

a) Find dim V.

b) Find $\mathrm{Hom}_{L_{R^+ \cup \{0\}}}(V, V)$.

c) Find $W_1, \ldots, W_t$ in V so that $V = \cup W_i$ where W_i's are semigroup super matrix semivector subspaces of V over $L_{R^+ \cup \{0\}}$ where $W_i \cap W_j = \phi$ if $i \neq j$ $1 \le i, j \le t$.

70. Find some nice applications of set super matrix semivector spaces defined over the set $S = L_{2^n \cup \{0\}} \subseteq L_{R^+ \cup \{0\}}$.

71. Does there exists a set super matrix semivector space which has no subset super matrix semivector subspace of refined labels? Justify your claim.

72. Does there exist a semigroup super matrix semivector space of refined labels which has no subsemigroup super matrix semivector subspace of refined labels.

73. Prove every integer set super matrix semivector space of refined labels has integer subset super matrix semivector subspace of refined labels!

74. Let $V = \left\{ \left[\begin{array}{c|ccc} L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ \hline L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \end{array}\right] \middle| L_{a_1} \in L_Q \right\}$ be a super matrix vector space of refined labels over the field L_Q.

a) Can V have non trivial super matrix vector subspace of refined labels?
b) Find $\text{Hom}_{L_Q}(V,V)$.
c) Does $T : V \to V$ exist so that $T^2 = T$? ($T \neq I$).
d) Find $L_{L_Q}(V, L_Q)$.

75. Let $V = \left\{ \left[\begin{array}{c|ccc|c} L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ \hline L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ \hline L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \end{array}\right] \middle| L_{a_1} \in L_{R^+ \cup \{0\}} \right\}$ be a semigroup super matrix semivector space of refined labels over the additive semigroup $L_{R^+ \cup \{0\}}$.

a) Find dim V.

b) Can V have non trivial semigroup submatrix semivector subspaces of refined labels over $L_{R^+ \cup \{0\}}$?

c) Can V have non trivial subsemigroup submatrix semivector subspaces of refined labels over subsemigroups in $L_{R^+ \cup \{0\}}$?

d) Find pseudo set super matrix semivector subspaces of refined labels over subsets in $L_{R^+ \cup \{0\}}$.

e) Find $\text{Hom}_{L_{R^+ \cup \{0\}}}(V, V)$.

76. Let $V = \left\{ \left[\begin{array}{c|ccc|c|cc} L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ \hline L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \end{array} \right] \middle| L_{a_1} \in L_R \right\}$

be a super matrix vector space of refined labels over L_R.

a) Can V have proper super matrix vector subspaces of refined labels over L_R?

b) Is V simple?

77. Let $V = \left\{ \left[\begin{array}{c|ccc} L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ \hline L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \end{array} \right] \middle| L_{a_1} \in L_Q \right\}$ be the super matrix vector space of refined labels over L_Q.

a) Is V simple?

b) Find $\text{Hom}_{L_Q}(V, L_Q)$.

78. Let $V = \left\{ \left[\begin{array}{c|cc} L_{a_1} & L_{a_1} & L_{a_1} \\ \hline L_{a_1} & L_{a_1} & L_{a_1} \\ L_{a_1} & L_{a_1} & L_{a_1} \end{array} \right] \middle| L_{a_1} \in L_{Q^+ \cup \{0\}} \right\}$

be a super matrix semivector space over the semifield $L_{Q^+ \cup \{0\}}$ of refined labels.

a) Find dim of V over $L_{Q^+ \cup \{0\}}$.

b) Is V simple?

c) Find $\text{Hom}_{L_{Q^+ \cup \{0\}}}(V, V)$.

79. Let V = $\left\{ \left[\begin{array}{c|ccc|c} L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ \hline L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ \hline L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \end{array} \right] \middle| L_{a_1} \in L_{R^+ \cup \{0\}} \right\}$ be a super matrix semivector space of refined labels over $L_{R^+ \cup \{0\}}$.

a) Is V simple?

b) Can V have substructures if instead of $L_{R^+ \cup \{0\}}$; V is defined over $L_{R^+ \cup \{0\}}$.

c) Find $\text{Hom}_{L_{R^+ \cup \{0\}}}(V, V)$.

d) Find T : V → V so that T^{-1} exist. (Is this possible).

e) Can T : V → V be such that T . T = T (T ≠ I)?

80. Let V = $\left\{ \left[\begin{array}{c|ccc} L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ \hline L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ \hline L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ \hline L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \end{array} \right] \middle| L_{a_1} \in L_{R^+ \cup \{0\}} \right\}$ be a group super matrix semivector spaces refined labels over the multiplicative group L_{R^+}.

a) Is V simple?

b) Does V have subgroup super matrix semivector subspaces?
c) Does V contain pseudo semigroup super matrix semivector subspaces?
d) Can V have pseudo set super matrix semivector subspaces of refined labels?
e) Find $\text{Hom}_{L_{R^+}}(V, V)$.

81. Characterize simple super matrix vector spaces of refined labels over L_R or L_Q.

82. Let V =

$$\left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_2} \\ L_{a_3} \\ L_{a_4} \\ L_{a_5} \\ \hline L_{a_6} \\ L_{a_7} \\ L_{a_8} \\ \hline L_{a_9} \\ L_{a_{10}} \\ \hline L_{a_{11}} \end{array}\right], \left[\begin{array}{c|c} L_{a_1} & L_{a_2} \\ \hline L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} \end{array}\right], \left[\begin{array}{ccc|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} & L_{a_6} \\ L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array}\right] \right|$$

$L_{a_i} \in L_{R^+ \cup \{0\}}; 1 \leq i \leq 12\}$ be a integer set super matrix semivector space of refined labels over $Z^+ \cup \{0\}$.

a) Find W_i's in V so that $V = \cup W_i$; $W_i \cap W_j = \phi$ if $i \neq j$.
b) Find $\text{Hom}_{Z^+ \cup \{0\}}(V, V)$.
c) Is V simple?
d) Is V finite dimensional?
e) Can V have subset integer super matrix semivector subspaces of refined labels over $S \subseteq Z^+ \cup \{0\}$?

83. Find the algebraic structure enjoyed by $\text{Hom}_{L_R}(V,V)$ where $V = \left(L_{a_1} \mid L_{a_1} \; \ldots \mid L_{a_1} \quad L_{a_1} \mid L_{a_1}\right); L_{a_1} \in L_R$.

84. Find the algebraic structure of $L_{L_R}(V, L_R)$ where V is given in problem 83.

85. Find the algebraic structure of where V =
$$\left\{\left.\left[\begin{array}{c|c|c} L_{a_1} & L_{a_1} & L_{a_1} \\ \hline L_{a_1} & L_{a_1} & L_{a_1} \end{array}\right]\right| L_{a_1} \in L_Q \right\}.$$

86. Find $\text{Hom}_{L_{Q^+}}(V,V)$ where V =
$$\left\{ \left[\begin{array}{c|c} L_{a_1} & L_{a_1} \\ \hline L_{a_1} & L_{a_1} \end{array}\right], \left(L_{a_1} \mid L_{a_1} \quad L_{a_1}\right), \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_1} \\ \hline L_{a_1} \end{array}\right], \left[\begin{array}{c|cc|cc} L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \\ L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} & L_{a_1} \end{array}\right] \right|$$
$\left. L_{a_1} \in L_{Q^+ \cup \{0\}} \right\}$ group super matrix semivector space of refined labels over the multiplicative group L_{Q^+}.

87. If L_{Q^+} is replaced by the semigroup $L_{Q^+ \cup \{0\}}$ under addition, V becomes the semigroup super matrix semivector space of refined labels over $L_{Q^+ \cup \{0\}}$; semigroup under addition.

Find $\text{Hom}_{L_{Q^+ \cup \{0\}}}(V,V)$.

a) Compare $\text{Hom}_{L_{Q^+ \cup \{0\}}}(V,V)$ in problem 86 with $\text{Hom}_{L_{Q^+ \cup \{0\}}}(V,V)$.

88. Let $V = \left\{ \left[\begin{array}{c} L_{a_1} \\ \hline L_{a_1} \\ L_{a_1} \\ \hline L_{a_1} \\ L_{a_1} \\ L_{a_1} \\ \hline L_{a_1} \\ L_{a_1} \\ L_{a_1} \\ L_{a_1} \end{array} \right] \middle| L_{a_1} \in L_R \right\}$ be a super matrix (column vector) vector space of refined labels over the field L_R.

a) Find $\text{Hom}_{L_R}(V,V)$.

b) $L_{L_R}(V, L_R)$.

89. Obtain some interesting properties enjoyed by $\text{Hom}_{L_Q}(V,V)$, V is a super row vector, vector space of refined labels over L_Q.

90. What are the applications of super square symmetric matrix vector space of refined labels over L_R?

91. What are the applications of super skew symmetric matrix vector space of refined labels over L_Q?

92. If V is a diagonal super matrix collection of refined labels vector space over L_R; can we speak of eigen values and eigen vectors of refined labels over L_R?

93. Can the theorem of diagonalization be adopted for super square matrix of vector space of refined labels over L_R?

94. Study super model of refined labels using the refined labels as attributes.

95. Let V = $\left\{\left[\begin{array}{c|c|c} L_{a_1} & L_{a_2} & L_{a_3} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} \\ \hline L_{a_7} & L_{a_8} & L_{a_9} \\ \hline L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \\ \hline L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \end{array}\right] \middle| L_{a_i} \in L_R; 1 \le i \le 15\right\}$ be a super matrix vector space of refined labels over L_Q.

a) Is V finite dimensional?
b) Does V have a subspace of finite dimension? Justify.

96. Obtain some interesting results related with hyperspace of super matrix vector space of refined labels.

97. Obtain some applications of super matrix semivector spaces of refined labels over $L_{R^+ \cup \{0\}}$ or $L_{Q^+ \cup \{0\}}$.

98. Prove L = $\left\{\left[\begin{array}{c|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ \hline L_{a_5} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ L_{a_6} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} \\ \hline L_{a_7} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \end{array}\right] \middle| L_{a_i} \in L_Q; 1 \le i \le 16\right\}$ is a group of super matrix of refined labels under addition.

99. Prove a super matrix vector space of refined labels over L_Q or L_R is never a super matrix linear algebra of refined labels over L_Q or L_R.

100. Let V = $\left\{\left[L_{a_1} \mid L_{a_2} \mid \ldots \mid L_{a_8}\right] \middle| L_{a_i} \in L_Q; 1 \le i \le 8\right\}$ be a super row matrix vector space of refined labels over L_Q.
V is of dimension 8.
How many super row matrix vector spaces of refined labels of dimension 8 can be constructed over L_Q?

101. If V = {$m \times n$ super matrix with entries from L_Q} be a super matrix vector space of dimension mn over L_Q.
How many such spaces be constructed with m natural rows and n natural columns?

102. Let T = $\begin{bmatrix} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ L_{a_6} & L_{a_{11}} & L_{a_{12}} & L_{a_{13}} \\ L_{a_7} & L_{a_{14}} & L_{a_{15}} & L_{a_{16}} \\ L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \end{bmatrix}$ be 5×4 matrix of refined labels. In how many ways can T be partitioned to form super matrix of refined labels.

103. Does the partition of a matrix of refined labels affect the basis algebraic structure?

104. If V = $\left\{ \left[\begin{array}{c|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_{10}} \\ \hline L_{a_4} & L_{a_5} & L_{a_6} & L_{a_{11}} \\ L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{12}} \end{array} \right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 12 \right\}$ and

$W = \left\{ \left[\begin{array}{c|cc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} \\ L_{a_5} & L_{a_6} & L_{a_7} & L_{a_8} \\ L_{a_7} & L_{a_{10}} & L_{a_{11}} & L_{a_{12}} \end{array} \right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 12 \right\}$ be two super matrix vector space of refined labels over L_R. Is $V \cong W$?

105. Let

$$V = \left\{ \left[\begin{array}{c|ccc|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ \hline L_{a_6} & L_{a_7} & L_{a_8} & L_{a_9} & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & L_{a_{15}} \\ L_{a_{16}} & L_{a_{17}} & L_{a_{18}} & L_{a_{19}} & L_{a_{20}} \end{array} \right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 20 \right\}$$

and

$$W = \left\{\left[\begin{array}{c|c|cc|c|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & \ldots & L_{a_{10}} \\ L_{a_{11}} & L_{a_{12}} & L_{a_{13}} & L_{a_{14}} & \ldots & L_{a_{20}} \end{array}\right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 20 \right\}$$

be two super matrix vector spaces of refined labels over L_R.

a) Is $V \cong W$?

b) Is dim V = dim W?

c) Is $\mathrm{Hom}_{L_R}(V, V) \cong \mathrm{Hom}_{L_R}(W, W)$?

106. Let $V = \left\{\left(L_{a_1} \mid L_{a_2} \;\; L_{a_3} \;\; L_{a_4} \mid L_{a_5}\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 5\right\}$

and $$W = \left\{\left(\begin{array}{c|cc|c|c} L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \\ L_{a_1} & L_{a_2} & L_{a_3} & L_{a_4} & L_{a_5} \end{array}\right) \middle| L_{a_i} \in L_R; 1 \leq i \leq 5\right\}$$

be two super matrix vector space of refined labels over L_R?

a) Is $V \cong W$?

b) Is $L_{L_R}(V, L_R) \cong L_{L_R}(W, L_R)$?

c) Is $\mathrm{Hom}_{L_R}(V, V) \cong \mathrm{Hom}_{L_R}(W, W)$?

107. Let $$V = \left\{\left[\begin{array}{c|ccc} L_{a_1} & L_{a_2} & 0 & L_{a_1} \\ \hline 0 & L_{a_3} & L_{a_4} & 0 \\ L_{a_2} & 0 & 0 & L_{a_1} \\ 0 & L_{a_4} & L_{a_3} & L_{a_2} \end{array}\right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 4\right\}$$ and W

$$= \left\{\left[\begin{array}{cc|cc|cc} L_{a_1} & 0 & L_{a_2} & 0 & L_{a_3} & 0 \\ 0 & L_{a_4} & 0 & L_{a_1} & 0 & L_{a_2} \\ L_{a_3} & 0 & L_{a_4} & 0 & L_{a_1} & 0 \\ \hline L_{a_1} & L_{a_3} & L_{a_2} & L_{a_4} & 0 & 0 \\ 0 & L_{a_4} & 0 & 0 & L_{a_1} & 0 \\ \hline L_{a_2} & 0 & L_{a_1} & L_{a_2} & 0 & L_{a_3} \end{array}\right] \middle| L_{a_i} \in L_R; 1 \leq i \leq 4\right\}$$

be two super matrix vector spaces of refined labels over L_R?

a) Is dim V = dim W?

b) Is $V \cong W$?

c) Find $\text{Hom}_{L_R}(V, W)$.

d) Is $\text{Hom}_{L_R}(V, V) \cong \text{Hom}_{L_R}(W, W)$?

e) Is $L_{L_R}(V, L_R) \cong L_{L_R}(W, L_R)$?

f) Find a basis for V and a basis for W.

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The authors in this book introduce the notion of DSm super vector space of refined labels. The notion of DSm semi super vector space of refined labels are also described. Several interesting properties are derived. We have suggested over 100 problems, some of which are research problems.

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